Inclusion-Exclusion

Sieve Formula

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Theorem

Let $A_1, \ldots, A_n$ be finite sets. Then,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right| .$$

Proof.

By induction on n. □

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Proof.
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1. Start proof with $n = 1$.
2. In the induction step, assume that the theorem holds for n sets $B_1, \ldots, B_n$.

For $n + 1$ sets $A_1, \ldots, A_{n+1}$ define

$$B_i := A_i \cup A_{n+1} \quad (i = 1, \ldots, n)$$

and apply the induction hypothesis to $B_1, \ldots, B_n$.

Leave the details as an exercise.

If there is enough time, mention the first occurrence of the principle of inclusion-exclusion in a letter from Bernoulli to Montmort in 1710.