Buoyancy oscillations

Bruno Voisin, LEGI, CNRS & Université Grenoble Alpes, France

December 7, 2021

A body immersed in a uniformly stratified fluid of buoyancy frequency N is displaced a vertical distance ζ_0 from its neutral buoyancy level $\zeta = 0$, then released at time $t = 0$. The body returns back to equilibrium through damped oscillations $\zeta(t)$. Their spectrum

$$\zeta(\omega) = \int_{-\infty}^{\infty} \zeta(t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t$$

is related to the vertical added mass coefficient $C_z(\omega)$ of the body (Voisin 2009) by

$$\frac{\zeta(\omega)}{\zeta_0} = \frac{\mathrm{i}}{\omega} \frac{1 + C_z(\omega)}{1 - N^2/\omega^2 + C_z(\omega)}.$$

We calculate the oscillations for an elliptic cylinder and a spheroid, based on expressions of their added mass deduced from Voisin (2021).

For the cylinder, of horizontal semi-axis a, vertical semi-axis b and aspect ratio $\epsilon = b/a$, we have

$$C_z = \frac{a}{b} \left(1 - \frac{N^2}{\omega^2}\right)^{1/2}.$$

The oscillation spectrum follows as

$$\frac{\zeta(\omega)}{\zeta_0} = \frac{\mathrm{i}}{(\omega^2 - N^2)^{1/2}} \frac{b\omega + a(\omega^2 - N^2)^{1/2}}{a\omega + b(\omega^2 - N^2)^{1/2}}.$$

Hereinafter, the determination of the multivalued functions of ω is set by causality, namely by the requirement that the functions be analytic in the upper half of the complex ω-plane. In particular, on the real axis, $(\omega^2 - N^2)^{1/2} = |\omega^2 - N^2|^{1/2} \operatorname{sign} \omega$ for $|\omega| > N$ and $\mathrm{i}|N^2 - \omega^2|^{1/2}$ for $|\omega| < N$.

The inverse transform

$$\frac{\zeta(t)}{\zeta_0} = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{(\omega^2 - N^2)^{1/2}} \frac{b\omega + a(\omega^2 - N^2)^{1/2}}{a\omega + b(\omega^2 - N^2)^{1/2}} \, \mathrm{d}\omega$$

is evaluated by the method of Larsen (1969, Appendix D). We change variable to

$$s = \frac{\omega}{N} + \left(\frac{\omega^2}{N^2} - 1\right)^{1/2},$$

such that $\omega = N(s + 1/s)/2$ and $(\omega^2 - N^2)^{1/2} = N(s - 1/s)/2$. The inverse transform becomes

$$\frac{\zeta(t)}{\zeta_0} = \frac{\mathrm{i}}{2\pi} \int_{\Gamma} \mathrm{e}^{-\mathrm{i}Nt(s+1/s)/2} \frac{(b+a)s^2 + (b-a)}{(b+a)s^2 - (b-a)} \frac{\mathrm{d}s}{s},$$

with Γ the contour in figure 1(a). The integrand has singularities $s = 0$ and $s^2 = (\epsilon - 1)/(\epsilon + 1)$, all inside the unit circle $|s| = 1$. The contour may thence be deformed to that in figure 1(b). The contributions of the two parts along the negative imaginary axis cancel out, and we are left with

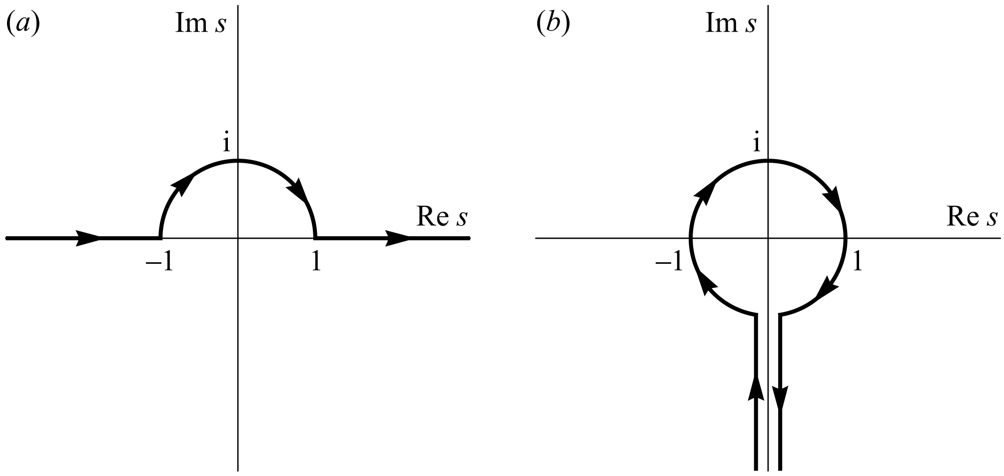

Figure 1: Contour of integration Γ in the s-plane, (a) before and (b) after deformation.

an integral along the unit circle. Changing variable again to $s = \mathrm{e}^{\mathrm{i}\theta}$, so that $\omega = N\cos\theta$ and $(\omega^2 - N^2)^{1/2} = \mathrm{i}N\sin\theta$, we obtain

$$\frac{\zeta(t)}{\zeta_0} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}Nt\cos\theta}\frac{b\cos\theta + \mathrm{i}a\sin\theta}{a\cos\theta + \mathrm{i}b\sin\theta}\,\mathrm{d}\theta.$$

This integral is further simplified by considering each quarter-circle $0 < \theta < \pi/2$, $\pi/2 < \theta < \pi$, $\pi < \theta < 3\pi/2$ and $3\pi/2 < \theta < 2\pi$ separately, changing variables so as to turn each integral into one over $0 < \theta < \pi/2$. Adding up the four contributions, we obtain

$$\frac{\zeta(t)}{\zeta_0} = \frac{2}{\pi}\int_0^{\pi/2} \cos(Nt\cos\theta)\frac{ab}{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta.$$

For a circular cylinder ($a = b$) the integral is a Bessel function

$$\frac{\zeta(t)}{\zeta_0} = \mathrm{J}_0(Nt),$$

and is otherwise evaluated numerically.

The spheroid has the added mass coefficient

$$C_Z = \left(1 - \frac{N^2}{\omega^2}\right)\frac{D(\Upsilon)}{1 - B(\Upsilon)},$$

where

$$\Upsilon = \frac{b}{a}\left(1 - \frac{N^2}{\omega^2}\right)^{1/2} \qquad \text{and} \qquad D(\Upsilon) = \frac{1}{1-\Upsilon^2}\left[1 - \frac{\Upsilon}{(1-\Upsilon^2)^{1/2}}\arccos\Upsilon\right].$$

The oscillation spectrum becomes

$$\frac{\zeta(\omega)}{\zeta_0} = \mathrm{i}\frac{\omega}{\omega^2 - N^2}\left[1 - \frac{N^2}{\omega^2}D(\Upsilon)\right].$$

Its inverse transform is evaluated as before, yielding

$$\frac{\zeta(t)}{\zeta_0} = -\frac{\mathrm{i}}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}Nt\cos\theta}\left\{\frac{(\epsilon^2-1)\tan\theta}{1+\epsilon^2\tan^2\theta} + \frac{\epsilon(1+\tan^2\theta)}{(1+\epsilon^2\tan^2\theta)^{3/2}}\left[\mathrm{i}\frac{\pi}{2} + \operatorname{arcsinh}(\epsilon\tan\theta)\right]\right\}\mathrm{d}\theta,$$

which simplifies to

$$\frac{\zeta(t)}{\zeta_0} = \int_0^{\pi/2} \cos(Nt\cos\theta)\frac{a^2 b\cos\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}\,\mathrm{d}\theta.$$

For a sphere this reduces to a Struve function

$$\frac{\zeta(t)}{\zeta_0} = \frac{\pi}{2}\mathbf{H}_{-1}(Nt) = 1 - \frac{\pi}{2}\mathbf{H}_1(Nt),$$

and is otherwise evaluated numerically.

The oscillations are plotted in figure 2. They decay faster for the two-dimensional cylinder than for the three-dimensional spheroid. When the body is streamlined, namely the spheroid is oblate or the cylinder elongated along the vertical, the oscillations are similar to those for a sphere or a circular cylinder, respectively, but their decay gets slower as the body gets more streamlined. When the spheroid is prolate or the cylinder elongated along the horizontal, the oscillations lose their near-symmetry with respect to the neutral buoyancy level. Instead, the oscillations may take several periods to reach this level and then, for spheroids of aspect ratios $\epsilon = 0.2$ and below, they overshoot it for several more periods before eventually coming back to it.

These conclusions, however, are unrealistic in two respects: first, by their neglect of viscosity, which plays a significant role in the damping of the oscillations; and second, by their neglect of the rotational motions of the oscillating body, especially significant for an oblate spheroid and a cylinder elongated along the horizontal, both of whose vertical oscillations are prone to rotational instabilities. This second neglect is inherent in the calculations of Voisin (2009, 2021), which only consider translational motion. Rotational motion was considered for an elliptic cylinder by Hurley & Hood (2001).

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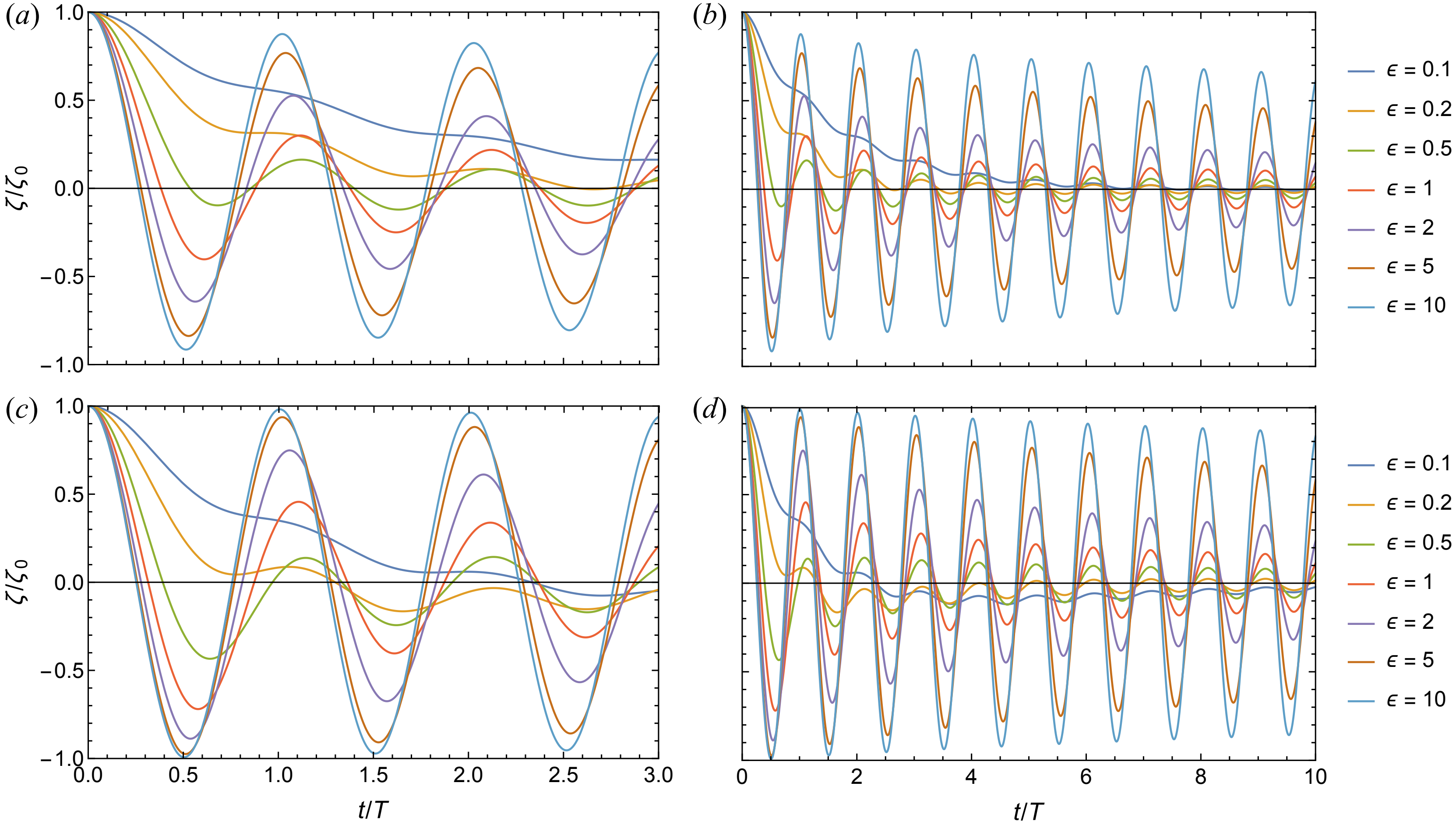

Figure 2: Buoyancy oscillations for (a,b) elliptic cylinders and (c,d) spheroids of varying aspect ratio ϵ versus non-dimensional time t/T, with $T = 2\pi/N$ the buoyancy period. Both (a,c) short- and (b,d) long-term evolutions of the oscillations are shown.