PHYSICS
for Scientists and Engineers

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Dedication

This book is dedicated to my wife, Mary Elaine Jones, and my son, David Wayne Jones.

Acknowledgements

The manuscript for this book would have never have been finished had it not been for the encouragement and support of family, friends and colleagues. I am especially grateful to Richard Logan, who followed its preparation through with many useful recommendations. The many others who have provided encouragement and support in this work before and after retirement are too many to mention but their contributions will always be appreciated.

Nothing in this book is original; all credit is due to the scientists who preceded me and who worked to uncover the principles of nature that we can discuss today.

Preface

Many approaches to physics are available to the modern student. Some are aimed at those students who plan to enter other disciplines and want to take with them only a passing familiarity with the principles of physics. Others are aimed at those students who plan for a career in science or engineering and want to take away a working knowledge of physics. This text is for those who plan for a professional career in science or engineering and requires a basic knowledge of differential and integral calculus for its successful completion.

Scope of material to be covered and time available for coverage are always competitors. The normal college course in physics must be covered in forty clock hours of contact instruction and tests spread over fourteen to fifteen weeks. This text covers the first two courses in what is commonly known as sophomore physics and is divided into ten chapters. Often textbooks in physics will cover over 1000 pages with hundreds of fragmented problems, which is entirely too long for careful reading and study in the time that can be allotted to one course. This text has been reduced in length to allow more time for study and thought on each subject. Problems have also been reduced in number but selected for more comprehensible coverage of the topic under consideration and application to problems encountered in everyday science and engineering.

The topics to be covered have been selected based on the author's lifetime experience in information needed in research and industry. Problems have been selected that address problems of the type often encountered in professional engineering and physics work. Topics addressed in this text have been organized to lead the student into more specialized and advanced courses in upper division undergraduate courses. Often, a source of reference material is needed in professional engineering and physics. For this purpose a number of tables and conversions factors have been included.

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List of Symbols

A area
a acceleration
α coefficient of linear expansion with temperature
β coefficient of bulk expansion
c speed of light
C_p molar specific heat at constant pressure
C_v molar specific heat at constant volume
c_p specific heat at constant pressure $\left(\frac{\partial H}{\partial T}_p\right)$
c_v specific heat at constant volume $\left(\frac{\partial H}{\partial T}_v\right)$
Δ change in....
δ deviation
E total energy
e electronic charge
ε emissivity factor for radiant heat transfer
ε mechanical strain
F mechanical force
g gravitational acceleration
G gravitational constant
H enthalpy
h elevation
J Joule
k thermal conductivity
K kinetic energy
L length
m mass
μ expectation value
N_o Avogadro's number
P power, probability, momentum, probably error
p pressure, momentum
Q heat
$\dot{Q}$ heat transfer per unit time
r radius, distance
S entropy

s	distance
σ	standard deviation
σ_m	standard error of the mean
T	temperature
t	time
U	internal energy
v	velocity, volume
W	work

Chapter 1

MEASUREMENT AND OBSERVATION

1 Concepts of Physics

1.1 Scientific Method

It may be said that the philosophy of science is to seek an understanding of natural phenomena by analysis. There are three fundamental steps by which natural phenomena can be analyzed. These include:

1. Measurement of physical quantities that can be observed,
2. Development of a theory by which these measurements can be explained and use of this theory to predict experimental outcomes,
3. Comparison of predicted results with measurement

This method, carefully followed, allows step by step progress in understanding nature. Progress may often be slow, and when discrepancies between predicted and measured outcomes are encountered we must return to the second step and start over. Mistakes and errors are almost always eliminated by this method.

1.2 Physical Quantities

The laws of physics are based on **physical quantities**, such as mass, length, velocity, energy, etc. Physical quantities may be defined as any observable and measurable quantity and divided into two categories. **Fundamental quantities** are those

quantities which are not based on other quantities. Examples are length, mass, time and charge. **Derived quantities** are quantities which may be derived from fundamental quantities. Examples are volume ($V = L^3$), velocity ($v = L/T$) and acceleration ($a = L/T^2$). Physical quantities may also be categorized according to their inherent properties. Those quantities which have both value and direction are called **vector quantities** while those quantities which possess only value but not direction are called **scalar quantities**.

1.3 Systems of Units

Measurements of physical quantities are expressed in terms of units which may be defined differently in different systems of measurement. The three most common systems of units are the British, Metric and Gaussian systems identified in the following table.

System	Length	Mass	Time	Force
British	foot	slug	second	Pound
Metric	meter	kilogram	second	Newton
Gaussian	centimeter	gram	second	Dyne

Table 1.1: Commonly used systems of units.

1.4 Standards

Having defined a system of units it was also necessary to define standards for the fundamental quantities. Any standard chosen had to be constant and readily available to all laboratories for calibration of instruments. In 1960 an international committee adopted the Metric system for the international system of units (SI) and established standards for fundamental units of mass, length, time, temperature, electric current and luminous intensity. The following current definitions of the base units are taken from NIST Special Publication 330, *The International System of Units*.

These units are adequate to address the needs of engineering and physics in the macroscopic world with which we are familiar, but problems arise when these macroscopic units are applied in the microscopic world of atomic, nuclear and

Base quantity	Name	Symbol	Definition
length	meter	m	distance traveled by light in 1/299792458 seconds
mass	kilogram	kg	1/12 the mass of the Carbon atom
time	second	s	9192631770 times frequency of hyperfine radiation from Cs^{133}
electric current	ampere	A	that current which if maintained in two parallel conductors would produce a force of 2 x 10^{-7} Newtons
temperature	kelvin	K	1/273.16 of the thermodynamic temperature of the triple point of water
amount of substance	mole	mol	amount of substance of a system which contains as many elementary entities as there are atoms
luminous intensity	candela	cd	the luminous intensity in a given direction of a source that emits monochromatic radiation of frequency 540 x 10^{12} hertz and has a radiant intensity in that direction of 1/683 watt per steradian

Table 1.2: Standards for fundamental quantities.

subnuclear particles. For example, the Bohr radius of the atom is 10^{-10} meters and the force between the electron and the proton is 10^{-8} Newtons. These results have ungainly powers of 10 that must be dealt with. Furthermore, these systems are based on arbitrary choices for standards of mass, length, time and charge and not on fundamental constants of nature. Scientists and engineers working in the first half of the 20th century therefore found it desirable to establish more fundamental units for use on the microscopic scale. While these systems of units are not important in this text, it is important to recognize their existence.[1]

[1]The first step toward eliminating cumbersome powers of 10 was made by choosing a system of units in which the mass and charge of the electron and the radius of the first Bohr orbit ($a_o = \hbar/m_o e^2 = 0.53x10^{-8}cm$) are taken as standards of mass, charge and length. This set of units is often called "atomic units" and is defined by $m_o = e = a_o = 1$. This system of units eliminates the cumbersome powers of 10 but it was designed with the atom in mind and not for universal application. It does not deal with the more fundamental problem of formulating a theory in terms of universal constants, and it includes no new standard for time. The second set of units, called the "natural system" was formed by defining the speed of light and Plank's constant $\hbar$ to be unity, $c = \hbar = 1$. These units have the advantage of being constant in nature and are useful in development of field theories, but they have the disadvantage that fundamental units of mass and charge become derived units and must be calculated.

1.5 Dimensional Analysis

Dimensional analysis is useful in assuring that derived formulas make physical sense and that the correct quantities are substituted for variables in the formulas. Dimensional analysis of a formula is performed by checking the dimensions of each term and requiring that the dimensions of each term in the formula be the same. For example, the dimensions of each term in the kinematic formula for the velocity of a particle experiencing a constant acceleration, $V = V_o + V_o Cos(\theta)t + \frac{1}{2}at^2$, must be dimensions of velocity (meters/second).

2 Reporting Single Measurements

2.1 Reporting Measurements

Measurement of any physical quantity is limited by the graduations on the measuring instrument and it is necessary to round off the result of the measurement to a value that is reliable. Consider, for example, measurement of a steel bar with a scale graduated in tenths of a centimeter as illustrated in figure 1.1. The length of the bar lies between 17.6 and 17.7 cm. The true length is clearly less than 17.65 cm, but it cannot be read directly.

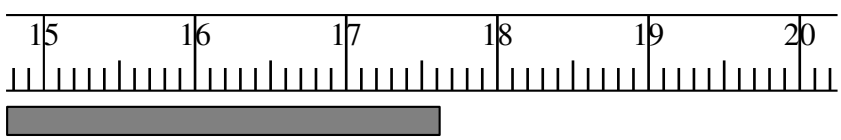


Figure 1.1: Measurement of bar with a scale graduated in tenths of a centimeter

A measurement should be rounded to the nearest graduation on the scale and, in this case, reported as 17.6 cm since the fraction is clearly less than one-half of the scale graduation. The same thing may be said of a measurement that lies between 17.55 and 17.6 so that measurement would also be reported as 17.6 cm. Therefore, a reported measurement of 17.6 cm must be interpreted to mean the length of the bar lies between 17.55 and 17.65 cm. This indicates an **uncertainty** in the length of the bar is $2\ x\ 0.05 = 0.1$ cm. The measurement could also be written as 17.6 ± 0.05 cm.

A general rule for rounding numbers is to drop the least accurate digit and raise the next digit by 1 if the digit dropped is more than one-half or retain the value of the next digit if the digit dropped is less than one-half. For example 17.63 would be rounded down to 17.6 while 17.68 would be rounded up to 17.7. When the digit to be dropped is exactly one-half, the often quoted rule is to round the next digit to an even number. For example, 17.65 would be rounded down to 17.6 while 17.75 would be rounded up to 17.8; but a prudent observer may consider reporting the half-unit as the best method when the results are to be used in mathematical calculations. This approach will be used in this text.

The number of reportable significant figures can be increased by adding a second scale known as a **vernier scale**. In a vernier scale, which slides along the main scale, the divisions are nine-tenths as long as the smallest divisions of the main scale. Hence, the ten divisions of the vernier scale have the same length as 9 divisions on the main scale and be read by noting the coincidence between the graduations of the vernier and main scale.

2.2 Scientific Notation

In reporting the result of a measurement, digits that are considered reliable are referred to as **significant figures**. The value of 17.6 represents three significant figures. The location of the decimal point has nothing to do with the number of significant figures and the value 17.6 will have only three significant figures whether reported as 176 millimeters, 0.176 meters or 17.6 centimeters. However, a measurement of 98 hundred meters would be incorrectly reported as 9800 meters if the measurement is reliable only to the nearest hundred meters. Including two zeros in 9800 implies that the measurement is reliable to 1 meter. Similarly, the value of 17.6 centimeters could not be reported as 1760000 microns since such a report would imply an accuracy of one micron (10^{-6}) meters.

To avoid misinterpretation, the results of measurements are best reported in **scientific notation**. In scientific notation, only significant figures multiplied by a power of ten are reported. For example, the value 17.6 centimeters can be reported as:

$$\begin{aligned} 17.6 &= 1.76\,x\,10^{-1} \quad \text{meters} \\ &= 1.76\,x\,10^{1} \quad \text{centimeters} \\ &= 1.76\,x\,10^{2} \quad \text{millimeters} \\ &= 1.76\,x\,10^{5} \quad \text{microns} \end{aligned}$$

2.3 Uncertainty and its Propagation

Uncertainty inherent in the report of a measurement will be propagated throughout calculations, often with a multiplying effect. Propagation of uncertainty is especially important when the value of a derived quantity depends upon successive calculations, some of which involve other derived quantities. The following general rules are often used to keep track of uncertainty during calculations. This discussion pertains to **single measurements**, and the following principles are useful in determining the **uncertainty** in results calculated from single measurements. It is important to realize that the uncertainty in a calculated result bears no relationship to the error in the result. Methods for computing error are discussed in subsequent sections.

To illustrate the uncertainties inherent in reported results, consider the case in which two measurements are reported as 43.9 and 3.45 meters. Based on the discussions in the preceding two sections, we can see that a number such as 43.9 may have been rounded from either 43.86 or 43.94, and a number such as 3.45 may have been rounded from either 3.446 or 3.454. The result of adding, subtracting, multiplying and dividing these two numbers is illustrated in table 1.3 with the results reported to 5 figures.[2]

Operation	Result	Minimum	Maximum
A	43.9	43.86	43.94
B	3.45	3.446	3.454
$A+B$	47.35	47.306	47.394
$A-B$	40.45	40.414	40.486
$A*B$	151.455	151 142	151.769
A/B	12.725	12.698	12.751

Table 1.3: The result of uncertainties in measurements on mathematical operations.

For the operations $A \pm B$, the exact result of the operation is reported in the results column. The extremes are reported in the next two columns. One way to report the results for addition and subtraction is $47.35(0.05)$ and $40.45(0.05)$ indicating an uncertainty of ± 0.05 in the last digit. For multiplication, the result could be reported as $151.455(0.315)$ indicating an uncertainty of ± 0.315 in the

[2]It must also be recognized that the engineer receiving the reports of measurements for use in his calculations will generally have no way to be certain what method of rounding the person making the measurements may have used.

last three digits and for division the result could be reported as $12.725(0.027)$ indicating an uncertainty of ± 0.027 in the last three digits. This method allows keeping the most information with the least uncertainty in the result of mathematical operations. **A commonly accepted method of reporting results in adding and subtracting is to round off the result to no more digits than appears after the decimal in the least exact number except when the calculated result contains the digit 5, which would result in reporting these measurements as 47.35 and 40.45. In multiplication and division the generally accepted rule is to round off the result to no more significant figures that appear in the least accurate factor. This method gives results of 151 and 12.7 for multiplication and division.** There is no universally accepted and used procedure for tabulating results subject to single measurement uncertainties.

Problems

1. Measurements of $A = 34.96$ cm and $B = 22.9$cm are reported. Report the sum, difference, product and quotient of A over B retaining the most information possible. ans. 57.860(0.044), 12.060(0.036), 800.58(1.49), 1.5266(0.0028)

2. One engineer reports the base of an isosceles triangle to be 50 cm while a second reports the height of the triangle to be 200 cm. What is the calculated, maximum and extreme values of the area that must be assumed by a third engineer who has only the reported values to go by? Report the results and the uncertainty in calculation. 5000(50)

3. A surveyor is measuring the distance across a stream of water. He lays out a course 50 ft long to form the perpendicular leg of a right triangle on one side of the stream and measures the angle subtended by the leg to be 30 degrees from the other side with an uncertainty of ± 1 degree. Assume the uncertainty in the perpendicular leg is 2 ft. Find the distance across the stream and the uncertainty in the result. ans. $86.6(+3.6/-3.4)$ feet

4. Suppose that the half-life of Cesium-137 is know to be 30.1 years with an uncertainty of 0.4 years and that a laboratory purchases a calibration source having an activity of exactly 1000 counts per second. What will the activity be one year later rounded to 4 significant figures? ans. $967.3(0.4)$ $counts/second$

3 Reporting Multiple Measurements

3.1 Errors

Measurement error as defined in this section is subject to a formal interpretation. It is important to distinguish between the uncertainty in a measurement, discussed in the preceding section as an uncertainty in what the actual result of the measurement may have been, and the error of the measurement. In the case of error we define:

- **Error** in a measurement is the difference between the reported result and the true value. Error is composed of both **systematic** and **random** effects and is not related to uncertainty.

- **Systematic** errors are introduced by a calibration of instruments, personal bias in measurement, unknown or uncontrollable external influence, etc. These errors generally contribute a fixed amount to a measurement. Usually, systematic errors can be identified and eliminated or a compensation developed to reduce their effect.

- **Random** errors are usually associated with a large number of uncontrollable factors that occur randomly and contribute both positive and negative amounts to the actual measurement. Random errors are difficult to control and almost impossible to eliminate but they can be defined and reported.

A good example of random and systematic errors is found in the shot pattern of a rifle fired from a machine rest at a target. The shot pattern, illustrated in figure 1.2, is grouped around a point displaced upward and to the right from the center. Displacement of the shot pattern corresponds to a systematic error. This systematic error is likely due to a misalignment of the gun sights and can be removed by correcting the misalignment. However, the scatter in the shots around the centroid of the shot pattern is due to random effects such as loose fittings, asymmetric effects in the bullets, a worn barrel, etc. These effects cannot normally be completely eliminated and represent true errors of measurement subject to the laws of chance.

A distinction between **accuracy** and **precision** should also be made. Accuracy is related to the error of a measurement and implies a reduction in systematic effects. The smaller the systematic error, the greater the accuracy. Precision is

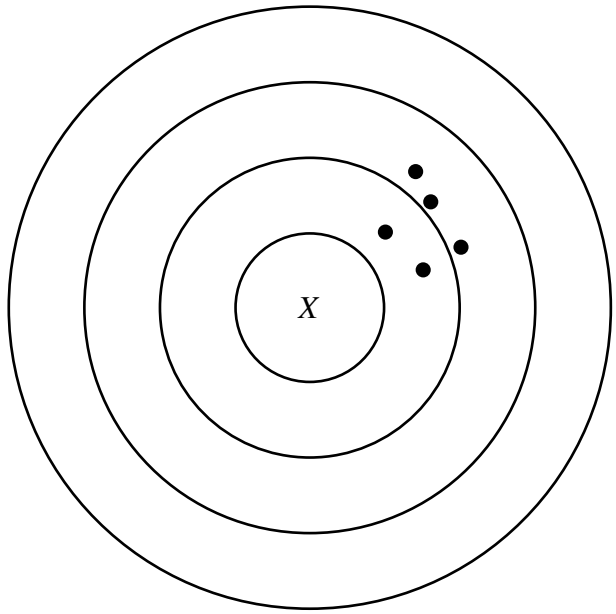


Figure 1.2: Shot pattern in bench test of rifle sights.

related to the spread in the results obtained from measurements of the same quantity and implies a reduction in random error. A precise measurement may still be subject to a large systematic error.

3.2 Error Management

The concepts and methodology discussed in section 2.3 dealt with reporting the results of single measurements and calculations involving these measurements. These concepts are important when only one measurement is available and no information is provided on the accuracy of the measurement. This is often the case in industrial work and engineering design when calculations must be based on a single measurement. In this section, we are concerned with analysis of the error when a large number of measurements of the same quantity can be made.

When the primary objective of an experiment is to measure the magnitude of a physical quantity, the measurement process must include elimination of systematic errors and a determination of random errors. Calibration of instruments will identify and reduce systematic errors. After systematic errors are identified, their contribution may be subtracted from the result which is to be reported. Examples of systematic errors include an offset in the zero point of calipers, the effect of

expansion with temperature on a metal yardstick, etc. There is usually a limit beyond which random errors cannot be reduced, but they can be identified and analyzed. After random errors have been determined, they can be reported along with the result of the measurement. These reported errors, or standard deviations, can be included in calculations in place of the uncertainty discussed in the preceding section.

As an example of measurements subject to random errors, consider the measurement of time for sound to travel between two points. Suppose that a stopwatch is started at the instant a gun is fired and stopped when the sound of the shot is reflected back from a cliff to the shooter. Data collected for such an experiment is listed for ten trials in Table 1.4 along with the deviation of each measurement from the arithmetic mean and the squares of the deviations. The variance and standard deviation of the mean are also calculated.

Trial No.	Time seconds (z)	Deviation seconds $\delta = z - \bar{z}$	$(Deviation)^2$ $(seconds)^2$ δ^2
1	1.84	-0.006	0.000036
2	1.86	0.014	0.000196
3	1.83	-0.016	0.000256
4	1.85	0.004	0.000016
5	1.87	0.024	0.000576
6	1.82	-0.026	0.000676
7	1.84	-0.026	0.000036
8	1.86	0.014	0.000196
9	1.85	0.004	0.000016
10	1.84	0.006	0.000036
Sum	18.46	0.000	0.002040
Average	1.846	Variance	0.000204
		Standard Deviation	0.0143
		Standard Error of the Mean	0.00452

Table 1.4: Measurements of speed of sound.

The average $\bar{z}$ of the measurements represents the **arithmetic mean** and is determined simply by dividing the sum of all measurements by the total number N of measurements. The **deviation** $\bar{\delta}_i$ of each measurement from the average, sometimes called the **residual**, is determined by subtracting the average $\bar{z}$ from

the measured value. Half of the residuals will be positive, half will be negative and the algebraic sum of all the residuals will be zero. The **average deviation** δ may be computed by summing the absolute values of all the residuals and dividing by the total number of measurements. The sum of the squares of the residuals divided by the total number of measurements is known as the **sample variance** V of the measurements. After the sample variance is determined, the **standard deviation** σ, **standard error of the mean** σ_m and **precision constant** h may be defined and computed as illustrated in table 1.5.

Average	$\bar{z} = \frac{\sum z_i}{N} = 1.846$
Residual	$\delta_i = z_i - \bar{z}$
Sample Variance	$V = \frac{\sum \delta_i^2}{N} = 0.000204$
Standard Deviation	$\sigma = \sqrt{V} = 0.0143$
Standard Error	$\sigma_m = \frac{\sigma}{\sqrt{N}} = 0.004517$
Precision constant	$h = \frac{0.7071}{\sigma} = 49.51$

Table 1.5: Formulas for calculating standard deviation and mean error

It should be noted that the sample variance computed by dividing the sum of the residuals squared by the total number of measurements N is an approximation of the true or theoretical variance. This discrepancy arises because the arithmetic mean $\bar{z}$ is only an approximation of the true expectation value of the measurements, which we will represent with μ. The sample variance computed by this formula gives too small a value for the variance. The true variance should be computed by dividing the sum of the residuals squared by $N-1$, which will produce a larger standard deviation and more accurately represent experimental results.

3.3 Normal Distribution

Since random errors are subject to the laws of chance their effect on the experiment can be made small by a large number of measurements. A formula representing the grouping of the measurements around an average value was derived by Karl Friedrich Gauss in 1795 and is known as the Gauss distribution curve, curve of error, or probability curve.

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2 z^2} \tag{3.1}$$

The ordinate of the curve illustrated in figure 1.3 represents the frequency of a given result for the hypothetical set of measurements in table 1.4. A mathematical analysis of this curve provides a methodology for describing analytically the error associated with a large number of measurements.

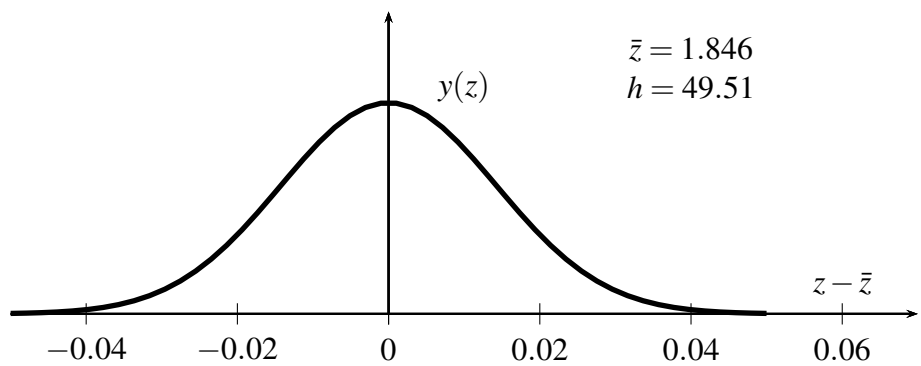


Figure 1.3: Gauss error curve.

The error curve shown in figure 1.3 is normalized to unit value so that the integral of $y(z)$ is unity. The constant h is known as the **precision constant** because it is a measure of the sharpness of the distribution curve. When h is large, the curve is sharp and $y(z)$ falls rapidly to the abscissa from the peak of the curve and measurements cluster closely around their mean value. When h is small, the curve is broad and $y(z)$ slowly approaches the abscissa. In this case, measurements are loosely cluster around their mean. The centroid of the curve is the expectation value of the measurements or true value of the quantity being measured. The expectation value is represented by μ, and defined by:

$$\mu = \frac{h}{\sqrt{\pi}} \int_a^b z e^{-h^2 z^2} dz \tag{3.2}$$

The probability of any given measurement falling within a range from $z = a$ to $z = b$ is represented by the integral of $y(z)$ from a to b.

$$P(a,b) = \frac{h}{\sqrt{\pi}} \int_a^b e^{-h^2 z^2} dz \tag{3.3}$$

The value of a and b for $P = \frac{1}{2}$ is called the **probable error** ε and may be computed from the distribution function as follows:

$$P(\varepsilon,\varepsilon) = \frac{h}{\sqrt{\pi}} \int_{-\varepsilon}^{\varepsilon} e^{-h^2 z^2} dz = \frac{1}{2} \tag{3.4}$$

to obtain

$$\varepsilon = \frac{0.4769}{h} \tag{3.5}$$

The **mean absolute error** is defined as the mean value of the deviation of each measurement.

$$\sigma_{abs} = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} |z| e^{-h^2 z^2} dx = \frac{0.5642}{h} \tag{3.6}$$

The mean square error or mean value of the square of the deviation, often called the **variance**, is computed from the integral of z^2 over the entire range of possible values of z.

$$\sigma^2 = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 e^{-h^2 x^2} dz = \frac{1}{2h^2} \tag{3.7}$$

to establish the **standard deviation**, $\sigma = \sqrt{\sigma^2}$ as

$$\sigma = \frac{1}{h\sqrt{2}} = \frac{0.7071}{h} \tag{3.8}$$

The probability that any measurement will fall between $\pm\sigma$ can be calculated from the probability curve

$$P_{-\sigma,\sigma} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\sigma}^{\sigma} e^{-\frac{z^2}{2\sigma^2}} dz = 0.6826895, \tag{3.9}$$

indicating that 68.3% of all measurements will fall within a range of $\pm\sigma$ on either side of the mean. In a similar manner,

$$P_{-2\sigma,2\sigma} = 0.9544997 \tag{3.10}$$
$$P_{-3\sigma,3\sigma} = 0.9973002 \tag{3.11}$$
$$P_{-4\sigma,4\sigma} = 0.9999366 \tag{3.12}$$
$$P_{-5\sigma,5\sigma} = 0.99999942 \tag{3.13}$$

indicating that 95.4% of all measurements fall within a range $\pm 2\sigma$, 99.7% fall within a range $\pm 3\sigma$, 99.994% fall within a range $\pm 4\sigma$ and 99.99994% measurements fall within $\pm 5\sigma$ of the mean.

3.4 Propagation of random error

The error resulting from the combination of two or more independent measurements will be larger than the errors of the individual measurements. It is important to have a method which will predict the probable value of this error. Random errors do not simply add arithmetically. Instead, the error in the result increases as the square root of the number of measurements. An understanding of why this is true can be had by assuming that a function u depends on two variables $u(x,y)$, taking the total differential and squaring it.

$$du = = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{3.14}$$

$$(du)^2 = \left(\frac{\partial u}{\partial x}\right)^2 (dx)^2 + \left(\frac{\partial u}{\partial y}\right)^2 (dy)^2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}dxdy \tag{3.15}$$

$$\tag{3.16}$$

If the measurements are uncorrelated, the third term will vanish so that the standard deviation of the error in $u(x,y)$ can be written as

$$du = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 (dx)^2 + \left(\frac{\partial u}{\partial y}\right)^2 (dy)^2} \tag{3.17}$$

Now, defining the standard deviation in $u(x,y)$ as $\sigma_u = \sqrt{(du)^2}$, and similarly for the standard deviation in x and y, a general formula for the error in the result of a mathematical combination of several measurements is described by:

$$\sigma_u = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2} \tag{3.18}$$

For example, in the case of a linear combination, $u = ax + by$ the standard deviation in u is

$$\sigma_u = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2}. \tag{3.19}$$

Suppose that the perimeter of a table of length ℓ and width w is measured with errors $\sigma_\ell = 2\ cm$ and $\sigma_w = 1cm$ in the measurements. The perimeter will be calculated from $P = 2\ell + 2w$ and the error will be

$$\sigma_P = \sqrt{(2)^2(2)^2 + (2)^2(1)^2} = \sqrt{16+4} = 4.47\ cm \tag{3.20}$$

The reasoning can be extended to other functions to obtain formulas for the propagation of error resulting from calculations. Examples are listed in table 3.4

$u = ax$	$\sigma_u = \sqrt{a^2\sigma_x^2}$
$u = ax + by$	$\sigma_u = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2}$
$u = axy$	$\frac{\sigma_u}{u} = \sqrt{\frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}}$
$u = a\frac{x}{y}$	$\frac{\sigma_u}{u} = \sqrt{\frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}}$
$u = ax^{\pm b}$	$\frac{\sigma_u}{u} = \frac{a\sigma_x}{x}$
$u = ae^{\pm bx}$	$\frac{\sigma_u}{u} = b\sigma_x$
$u = aln(\pm bx)$	$\sigma_u = a\frac{\sigma_x}{x}$

Table 1.6: Propagation of error formulas.

Problems

5. A marine scientist measures the speed of sound in sea water in 10 trials obtaining values of 1548, 1563, 1559, 1576, 1585, 1576, 1572, 1548, 1596 and 1582 $meters/second$. Calculate the standard deviation and standard error of one measurement and the precision constant of the measurements and plot a Gaussian curve for the measurements. ans. $\sigma = 82.13\ m/sec$, $\sigma_m = 25.97\ m/sec$, $h = 0.008609\ seconds/meter$

6. The quantity u is defined by $u = 5x + 8y$. Ten measurements of x yield 35, 36, 34, 38, 37, 39, 35, 36, 38 and 35 while ten measurements of y yield 20, 22, 21, 23, 25, 26, 21, 20, 22 and 23. Calculate u and the standard deviation in x, y and u and report using 2 significant figures. ans. 1.6, 1.9, $3.6x10^2$, $1.7x10^1$

7. Repeat the last problem for the case where $u = 5xy$ and report using 2 significant figures. ans. $4.1x10^3$ and $3.9x10^2$

4 Method of Least Squares

Often, the value of a physical variable is measured as a function of another independent variable. In many cases it is important to determine a relationship between the dependent and independent variables. For example, the amount of of sodium chloride which will dissolve in water at temperature T^oC is reported in table 1.7.

Temp T	Density d	T^2	Td
0	275	0	0
10	322	100	3220
20	334	400	6680
30	385	900	11550
40	396	1600	15840
50	416	2500	20800
60	457	3600	27420
70	485	4900	33950
80	523	6400	41840
90	538	8100	48420
100	571	10000	57100
550	4702	38500	266820

Table 1.7: Solubility of Sodium Chloride in water.

It would be useful if a straight line $d = a + bT$ could be fitted to these points so that solubilities between points where no measurements were taken could be calculated. This line can be found by constructing the sum of the squares of the deviations of points on this theoretical line $a + bT_i$ from the measured solubilities d_i.

$$S = [d_1 - (a + bT_1)]^2 + [d_2 - (a + bT_2)]^2 + \cdots + [d_{10} - (a + bT_{10})]^2 \tag{4.1}$$

The values of a and b that minimize the summation can be found by differentiating the quantity S using the Chain Rule and setting the partial derivatives with respect to a and b equal to zero. From the resulting equations, we obtain the following relationships between the quantities a and b.

$$na + (\sum T)b = \sum d \tag{4.2}$$

$$(\sum T)a + (\sum T^2)b = \sum Td \tag{4.3}$$

Solving these equations for a and b results in

$$a = \frac{N_a}{D} \tag{4.4}$$

$$b = \frac{N_b}{D} \tag{4.5}$$

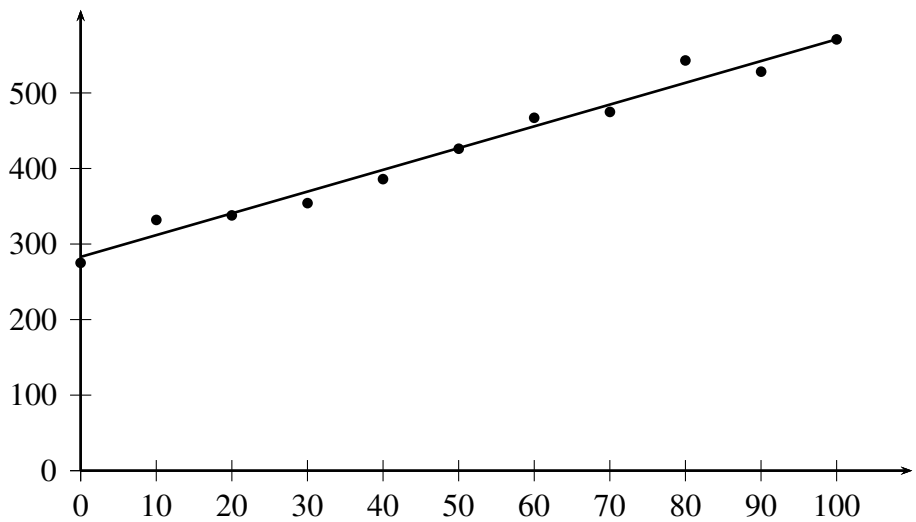


Figure 1.4: Solubility of Sodium Chloride in water with fitted curve.

where

$$D = \begin{vmatrix} n & \sum T \\ \sum T & \sum T^2 \end{vmatrix}, \quad N_a = \begin{vmatrix} \sum d & \sum T \\ \sum Td & \sum T^2 \end{vmatrix}, \quad N_b = \begin{vmatrix} n & \sum d \\ \sum T & \sum Td \end{vmatrix} \tag{4.6}$$

Using these formulas, we have

$$D = \begin{vmatrix} 11 & 550 \\ 550 & 38500 \end{vmatrix} = 121,000 \tag{4.7}$$

$$N_a = \begin{vmatrix} 4702 & 550 \\ 266820 & 38500 \end{vmatrix} = 34,276,000 \tag{4.8}$$

$$N_b = \begin{vmatrix} 11 & 4702 \\ 550 & 266820 \end{vmatrix} = 348,920 \tag{4.9}$$

$$\tag{4.10}$$

so that

$$a = \frac{34276000}{121000} = 283.3 \tag{4.11}$$

$$b = \frac{348920}{121000} = 2.88 \tag{4.12}$$

As a result the equation of the best line, plotted in figure 1.4, passing through the data points becomes

$$d = 283.3 + 2.88T \tag{4.13}$$

Problems

8. The scientist finds that the speed of sound increases with depth below the surface obtaining values of 1480, 1495, 1505, 1519, 1526, 1531, 1548, 1550, 1560 and 1570 meters/second for each 500 meters of depth. Using the method of least squares, find an equation that describes the increase in the speed of sound with depth in water. ans $V = 1485 + 0.1917d$

9. A isotope produced in a reactor is found to have an initial count rate of 490 $counts/second$ in a one-second count 25 seconds after removal and then counts of 424, 368, 319, 276, 240, 208, 180, 156, 135, 117, 102, 88 and 76 per second at subsequent intervals of 5 seconds. Given that the isotope decays at a rate given by the formula $I = I_o exp(-\lambda t)$, plot the data on log paper, perform a least squares analysis to find the decay constant λ and calculate the initial activity I_o ans. $I_o = 1002$ and $\lambda = 0.02862$

References

1. Y. Beers, *Introduction to the Theory of error*, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1957.

2. P. R. Bevington, *Data reduction and error Analysis for the Physical Sciences*, McGraw-Hill Book company, New York, NY., 1969.

Chapter 2

NON-ROTATING MOTION

Mechanical systems may be either **non-rotating** or **rotating**. For a complete understanding of mechanical systems, we must develop a means of describing their motion, which we will call **kinematics**, and a means of describing the effect of forces on the system, which we will call **dynamics**. We will begin by deriving the kinematic and dynamic relationships used to describe non-rotating systems in this chapter. To obtain these kinematic and dynamic relations in a non-rotating system we will focus our attention on the motion of a single particle in one, two and three dimensions. A study of rotating systems will be undertaken in the next chapter, which will also include internal motion of a body.

5 Motion along a trajectory in space

Consider the example illustrated in figure 5 where a particle of mass m traces out a trajectory at the end of a vector $\vec{\mathbf{R}}$ which identifies the location of the particle at any point in time. The vector $\vec{\mathbf{R}}$ defining the position of the particle in three dimensions may be thought of as a function of the position coordinates (x, y and z) along each axis of a three-dimensional coordinate system. In case the motion of the particle is without constraint, the particle will have three degrees of freedom.

$$\vec{\mathbf{R}}(x,y,z) = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y + \hat{\mathbf{k}}z \tag{5.1}$$

In a physical system, however, there is usually a constraint relating the value of one coordinate to another thus reducing the number of degrees of freedom of the particle. In a two-dimensional system, for example, consider a bead moving around a circle or an ellipse, along a parabola or the more complicated case of a

particle moving along a catenary between two posts. The constraining equations, usually referred to in mathematics texts as **rectangular equations**, are illustrated in Table 2.1 for each case.

In other systems, there may be no physical constraint between the coordinates, but the magnitude of each position coordinate may be a function of another variable, which we can consider a parameter and write the rectangular equations in **parametric form**. When they exist, the three equations, $x = x(t)$, $y = y(t)$ and $z = z(t)$ can be thought of as parametric representations of a curve in space with t representing the **parameter**, and the position of any point in space can be expressed as follows:

$$\vec{\mathbf{R}}(t) = \hat{\mathbf{i}}x(t) + \hat{\mathbf{j}}y(t) + \hat{\mathbf{k}}z(t) \tag{5.2}$$

The parametric equations are important because a physical problem is often defined in terms of the parametric equations which then form the starting point for the problem. For example, the initial conditions defining the coordinates x and y for a projectile launched into the air with a velocity V_o at an angle θ with the horizontal are $x = V_o t\cos\theta$ and $y = V_o t\sin\theta - \frac{1}{2}gt^2$, where t is the time of flight. Also, it is often easier to plot the algebraic curve by calculating the coordinates of each point from the parametric equations than using the rectangular equations.

When the initial starting point is the parametric equations and the rectangular equation is desired, it can be obtained from the parametric equations simply by eliminating the parameter between the equations. If, on the other hand, the problem is defined with the rectangular equation the parametric equations can be obtained by making an arbitrary assumption to define one coordinate in terms of a parameter that appears useful, inserting the definition into the rectangular equation and solving for the other variable.

Object	Rectangular Equation	Parametric Equations	F(x,y)
Circle	$x^2+y^2=a^2$	$x=a\cos\theta$ $y=a\sin\theta$	$F(x,y)=x^2+y^2-a^2$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x=a\cos\theta$ $y=b\sin\theta$	$F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$
Parabola	$y^2=x$	$y=t$ $t^2=x$	$F(x,y)=x-y^2$
Parabola	$y=\frac{b}{a}x-\frac{c}{a^2}x^2$	$x=at$ $y=bt-ct^2$	$F(x,y)=\frac{b}{a}x-\frac{c}{a^2}x^2-y$
Catenary	$y=\frac{a}{2}\left(e^{(x/a)}+e^{-(x/a)}\right)$	$x=a\ln\sec\phi+\tan\phi$ $y=a\sec\phi$	$F(x,y)=\frac{a}{2}\left(e^{(x/a)}+e^{-(x/a)}\right)-y$

Table 2.1: Parametric equations for various figures.

Using the parametric representation of the position vector, we can calculate the **velocity** of the particle from the first derivative of the position vector $\vec{\mathbf{R}}(t)$. The velocity vector will always be tangent to the path of the trajectory with a magnitude equal to the speed with which the particle is moving.

$$\vec{\mathbf{V}}(t) = \frac{d\vec{\mathbf{R}}}{dt} \tag{5.3}$$

The **acceleration** of the body can be represented as the first derivative of its velocity.

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{V}}}{dt} \tag{5.4}$$

Equations 5.3 and 5.4 provide the magnitude and direction of the velocity and acceleration vectors of the point in space. To plot these vectors on the graph, we must take recourse to the traditional methods of analytic geometry. Using these methods, the velocity vector passing through the point $(x_o, y_o z_o)$ can be represented as:

$$F_x(x_o, y_o, z_o)\,(x - x_o) + F_y(x_o, y_o, z_o)\,(y - y_o) + F_z(x_o, y_o, z_o)\,(z - z_o) = 0 \tag{5.5}$$

$$\text{where} \quad F_x(x_o, y_o, z_o) = \frac{\partial F(x_o, y_o z_o)}{\partial x} \quad \text{etc.} \tag{5.6}$$

To illustrate the application of these formulas in a practical problem, we choose the second parabola in Table 2.1 and set the parameters $a = 52$, $b = 193$ and $c = 16$. Inserting these values in the rectangular equation for the parabola gives the equation of motion and trajectory of the particle in physical space.

$$y = 3.71x - 0.00591x^2 \tag{5.7}$$

Using the values in the parametric equations gives the equations for position of the projectile in physical space, its velocity and acceleration as a function of time.

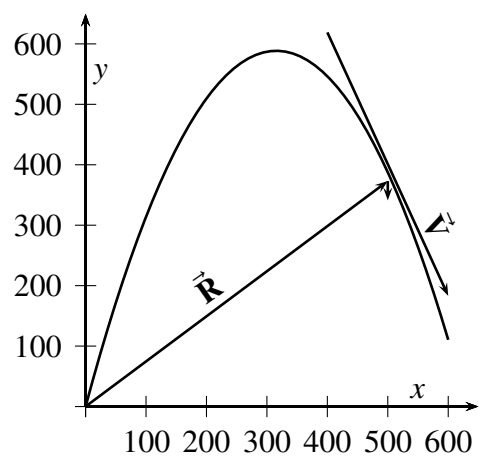


Figure 2.1: Projectile motion.

$$\vec{\mathbf{R}}(t) = 52\hat{\mathbf{i}}t + \hat{\mathbf{j}}\left(193t - 16t^2\right) \tag{5.8}$$

$$\vec{\mathbf{V}}(t) = 52\hat{\mathbf{i}} + \hat{\mathbf{j}}\,(193 - 32t) \tag{5.9}$$

$$\vec{\mathbf{a}}(t) = -32\hat{\mathbf{j}} \tag{5.10}$$

The vector representing acceleration always has a constant magnitude and points vertically downward. The vector representing the velocity varies in length and direction as the projectile moves along its trajectory and requires an additional equation for plotting on a graph. The velocity vector is always tangent to the path of motion. At the point $X_o = 500$ and $Y_o = 373$ identified by the parameter $t = 9.66$ the equation for the velocity is obtained from equation 5.5 and given by:[1]

$$y = 1492 - 2.24x \tag{5.11}$$

Problems

10. A bead moves along a parabola described by the rectangular equation $y = 8x^2$. Write equations giving the position, velocity and acceleration of the bead as a function of x and y and the equation for the tangent line through the point (2,32). Obtain parametric equations, calculate each point and plot a graph of the rectangular equation and the tangent line. ans. $\vec{\mathbf{R}} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}8x^2$, $\vec{\mathbf{v}} = \hat{\mathbf{i}}\dot{x} + 16\hat{\mathbf{j}}x\dot{x}$, $\vec{\mathbf{a}} = \hat{\mathbf{i}}\ddot{x} + 16\hat{\mathbf{j}}\left[x\ddot{x} - (\dot{x})^2\right]$, y=-32+32x.

11. Obtain the equation for the tangent line to a circle described by the rectangular equation, $x^2 + y^2 = 4$, at the point (x_o, y_o) obtained when the parameter $\theta = 45^o$ for the parametric equation and plot on a graph. ans. $y = 5.656 - x$

12. A catenary defined by the rectangular equation $y = 3\left(e^{x/6} + e^{-x/6}\right)$. Obtain the parametric equations and the equation for the tangent line to at the point $x = 3$ and plot on a graph. ans. $y = 5.203 + 0.521x$

6 Dynamics

In this section the concepts necessary to a study of the motion of objects in space and changes in that motion will be examined. These concepts include force, acceleration, momentum, work, power, kinetic energy, potential energy and total energy.

[1] These equations fully characterize the motion of a projectile through space when fired with a velocity of 200 m/sec at an angle of 75° and will be revisited in Section 10.

6.1 Force and Acceleration

Force and acceleration are related by what is usually taken to be the most simple formula of physics. This law of physics is attributed to Sir Isaac Newton (1643 – 1727), an English natural philosopher. In his book published in 1687, Newton presented three physical laws to describe the relationship between the forces acting on a body and its motion due to those forces. These laws are generally considered to be the basis for classical mechanics.[2]

1. **First Law** All bodies remain in a state of rest or uniform motion at constant velocity unless acted upon by an external unbalanced force.

2. **Second Law** A body of mass m subject to a force F undergoes an acceleration in the same direction as the force and with a magnitude directly proportional to the force and inversely proportional to its mass. The second law is often embodied mathematically as follows

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \tag{6.1}$$

3. **Third Law** When acted upon by a force F, a body exerts and equal and opposite force upon the body exerting the force.

Newton's second law usually carries the title of "Newton's Law" and is the basis of most calculations in dynamics. Use of Newton's law must be consistent with the system of units in table 1.1. In the SI system, for example, a 2 kilogram mass to which a force of 10 Newtons is applied will accelerate at a rate of 5 m/sec^2. Similarly, in the British system, a mass of 5 slugs will accelerate at a rate of 10 ft/sec^2 when a force of 50 pounds is applied.[3]

The first law can be considered a corollary of the second. If there are no forces acting on the mass, the acceleration is zero and the velocity is therefore constant. The third law will be established through conservation laws later in this text.

[2] Issac Newton, "PhilosophiÃŁ Naturalis Principia Mathematica" July 5, 1687

[3] It is important that students new to the study of physics develop a good grasp of these systems of units. Most students will be familiar with the use of the word "kilogram" as a unit of mass and the use of the word "pound" as a unit of force; but few will be familiar with the use of the word "Newton" as a unit of force and few will have heard of the term "slugs" or its use as a unit of mass or that the units cannot be mixed in calculations.

6.2 Momentum and Kinetic Energy

Two inherent properties of a body of mass m and velocity $\vec{\mathbf{V}}$ are its momentum and kinetic energy. The **momentum** is a vector quantity defined by the product of the mass m and velocity $\vec{\mathbf{V}}$.

$$\vec{\mathbf{p}} = m\vec{\mathbf{V}} \tag{6.2}$$

The mass and velocity of the body can also be combined to define the **kinetic energy** K, a scalar quantity without direction.

$$K = \frac{1}{2}mV^2 = \frac{p^2}{2m} \tag{6.3}$$

It may also be noted from equation 6.1 that the force acting on an object may be expressed as the time rate of change of the momentum.

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = m\frac{d\vec{\mathbf{V}}}{dt} = m\vec{\mathbf{a}} \tag{6.4}$$

From this equation it is seen that if the sum of all the external forces is zero, then the momentum is conserved and is a constant of the motion.

6.3 Work and Power

Consider the particle of mass m moving along the path indicated in figure 2.2 under the influence of a force $\vec{\mathbf{F}}$ from A to B. By definition, the work done by a force $\vec{\mathbf{F}}$ in moving a particle of mass m along a path C is defined by

$$W = \int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{6.5}$$

The unit of work defined by equation 6.5 is the Newton-meter, or **Joule**, in the SI system; the dyne-cm, or **erg**, in the cgs system; and the **foot pound** in the British system. Conversions are 1 Joule = 10^7 ergs = 0.7376 ft-lb. The British thermal unit, or **Btu**, also in common use equals 777.9 ft-lb.

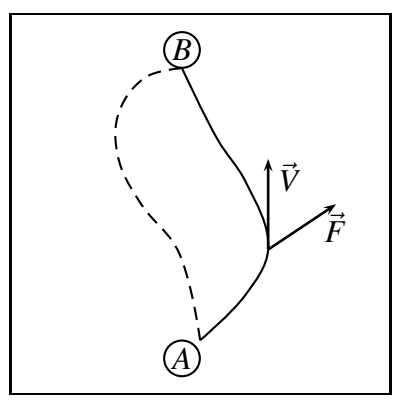


Figure 2.2: Mechanical system with a body of mass m moving in a plane.

Equations 6.4 and 6.5 can be combined to obtain the rate at which work is done, or **power**, defined by the equation

$$P = \frac{dW}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{V}} \tag{6.6}$$

The unit of power is the $Joule/sec$, or **Watt**, in the SI system; the erg/sec in the cgs system; and the $ft-lb/sec$ in the British system. The $Btu/hr = 0.2161\ ft-lb/sec$ is also used in the British system. Another unit in common use is the **Horsepower**, defined as the work output that a large horse was believed to be able to sustain indefinitely. One horsepower is equivalent to 745.7 watts, 550 $ft-lb/sec$ and 2545 Btu/hr.

Because of the widespread use of the Metric and British systems, it is often necessary to convert results in one system to another. To facilitate these conversions a table of conversions is provided in Appendix B. The historical development of each of these systems makes conversion factors ungainly but at least the definitions of terms in each are such that one simple rule allows easy conversions in heat transfer problems.

$$1cal/g/^oC = 1kcal/kg/^oC = 1Btu/lb/^oF = 1Btu/lb/^oR \tag{6.7}$$

6.4 Conservative Systems

Let us now suppose that the force $\vec{\mathbf{F}}$ is not constant but instead a function of its position at any point and that the force can be written as the gradient of a scalar potential function $U(r)$.

$$\vec{\mathbf{F}} = \nabla U(r) = \hat{\mathbf{i}}\frac{dU}{dx} + \hat{\mathbf{j}}\frac{dU}{dx} + \hat{\mathbf{k}}\frac{dU}{dx} \tag{6.8}$$

We can now evaluate the line integral to obtain the work done on the particle by this force in going from points A to point B.

$$\begin{aligned} W &= \int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_A^B \left(\hat{\mathbf{i}}\frac{dU}{dx} + \hat{\mathbf{j}}\frac{dU}{dy} + \hat{\mathbf{k}}\frac{dU}{dz}\right) \cdot \left(\hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy + \hat{\mathbf{k}}dz\right) && (6.9) \\ &= \frac{1}{2}\left(\frac{dU}{dx}dx + \frac{dU}{dy}dy + \frac{dU}{dz}dz\right) && (6.10) \\ &= U(B) - U(A) && (6.11) \end{aligned}$$

This equation tells us that the work done by the force which is a function of position in moving the particle from points A to B is equal to the change in the

the potential function $U(r)$ and independent of the path taken from points A to B. This prompts us to define the function $U(r)$ as the **potential energy**. Since the line integral depends only on the end points, it is also clear that as the particle moves around a closed loop to return to the starting point the total work done by the force will be zero. When the integral around a closed path is zero, the force is said to be **conservative**.

$$W = \oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0 \tag{6.12}$$

In summary, it may be said that the validity of either of the following three conditions is necessary and sufficient to guarantee the validity of the other conditions.

1. The force is the gradient of a uniform scalar function. $\vec{\mathbf{F}} = \nabla U(r)$
2. The work done by the force in moving a particle from one point to another depends only on the end points. $\int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = U(B) - U(A)$
3. The work done by the force in moving the particle around a closed path is zero. $\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$

To complete the definition of conservative force fields, the following point should be added although it will not be needed for this discussion. If $\vec{\mathbf{F}}(r)$ is a continuously differentiable vector force and S is a regular open surface the edge of which is bounded by a curve C around which the integral $\oint \vec{\mathbf{F}}(r) \cdot d\vec{\mathbf{r}}$ is zero, we know from Stokes theorem that the curl of the force vector $\vec{\mathbf{F}}(r)$ must be zero. Hence requiring that the curl of the force vector vanish is equivalent to requiring the validity of the above three conditions and that the force field is conservative.

$$\nabla \times \vec{\mathbf{F}}(r) = 0 \tag{6.13}$$

6.5 Total Energy and its Conservation

The sum of the kinetic and potential energies is the **total energy** of a system.

$$E = K + U \tag{6.14}$$

In a system acted on only by conservative forces, the total energy remains constant. This theorem can be established by an examination of the equation of

motion for a particle under the action of a conservative force. Writing the velocity $\vec{\mathbf{V}}$ as $\dot{\vec{\mathbf{r}}}$, we have

$$\vec{\mathbf{F}} = \frac{d}{dt} m\dot{\vec{\mathbf{r}}} \tag{6.15}$$

Taking the dot product of $\dot{\vec{\mathbf{r}}}$ with the left and right sides and noting that $\dot{\vec{\mathbf{r}}} \cdot \dot{\vec{\mathbf{r}}} = \frac{1}{2}\frac{d(\dot{\vec{\mathbf{r}}}\cdot\dot{\vec{\mathbf{r}}})}{dt} = \frac{1}{2}\frac{dV^2}{dt}$ gives.

$$\dot{\vec{\mathbf{r}}} \cdot \vec{\mathbf{F}} = \dot{\vec{\mathbf{r}}} \cdot \frac{d}{dt} m\dot{\vec{\mathbf{r}}} \tag{6.16}$$

$$\vec{\mathbf{F}} \cdot \frac{d\mathbf{r}}{dt} = m\vec{\mathbf{V}} \cdot \frac{d\vec{\mathbf{V}}}{dt} \tag{6.17}$$

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = mVdV \tag{6.18}$$

so that each side may be evaluated to get

$$U_B - U_A = \frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2 \tag{6.19}$$

Using the formula for kinetic energy, $K = \frac{1}{2}mV^2$ the result may be rewritten.

$$E = K_A + U_A = K_B + U_B \tag{6.20}$$

Thus the sum of the kinetic and potential energies of a particle acted on only by conservative forces remains constant throughout the motion. As a result we can say that the total energy is conserved when the only forces acting on the system are conservative. Today, this theorem is considered one of the most important principles of physics.

As a historical note, the concept of energy equivalence was not understood until late into the 19th century, after it was established independently by Julius von Mayer (1814-1878) in Germany, James Joule (1818-1889) in England, Hermann von Helmholtz (1821-1894) in Germany, L. A. Colding (1815-1888) in Denmark and Sadi Carnot (1796-1832) in France. Joule was the first to show by experiment the equivalence of mechanical work and heat while Helmholtz was the first to state the equivalence of all forms of energy.

Problems

13. What acceleration would a force of 10 N acting on a mass of 2 kg produce? Assuming the mass begins motion from rest, what is the velocity, momentum and kinetic energy after 5 seconds of motion: ans. 5 m/sec^2, 25 m/sec, $50kg - m/sec$, 625 joules

14. An elastic rubber ball of mass 0.1 kg moving a velocity of 10 m/sec strikes a wall and rebounds with the same velocity. What is the change in momentum during the collision and kinetic energy of the ball before and after rebound? ans. 2 $kg-m/sec$, 5 joules, same

15. Prove that the force $F = kx$ meets the three conditions for a conservative force.

7 Gravity

Included with his three laws of force, Sir Issac Newton proposed a universal law of gravitation in his *PhilosophiÃŁ Naturalis Principia Mathematica* to describe the gravitational force between two bodies of masses M_1 and M_2 located a distance $\vec{r}$ apart. Written in mathematical terms, this law is

$$\vec{F} = -G\frac{M_1 M_2}{r^2}\hat{r}. \tag{7.1}$$

Newton's law of gravity is a good illustration of his third law of force. Each of the masses M_1 and M_2 exerts and equal and opposite force on the other mass. In this formula, the constant G is known as the **gravitational constant** and the negative sign signifies the gravitational force is attrative. The first accurate measurement of the gravitational constant was made by Lord Cavendish in 1798. The currently accepted value is[4]

$$G = 6.67384(80)x10^{-11} \quad \frac{Newton-Meter^2}{kilograms^2} \tag{7.2}$$

$$= 3.435773x10^{-8} \quad \frac{Pound-Foot^2}{Slug^2} \tag{7.3}$$

The gravitational force is present between all bodies having mass. Examples of this force for large and small systems is illustrated in table 2.2

Since there is no medium connecting the two bodies the concept of a force field was developed. In this case, the force field is called a **gravitational field**. One object establishes this field and the field acts on the other object so that the

[4] "CODATA Recommended Values of the Fundamental Physical Constants: 2010". Committee on Data for Science and Technology (CODATA). NIST

Proton and electron in H-atom	3.63×10^{-47} Newtons
Two 10 kg lead balls 1 meter apart	6.67×10^{-8} Newtons
Earth and Moon	2.03×10^{39} Newtons

Table 2.2: Comparison of gravitational force for large and small objects

gravitational force may be divided between two equations, one of which represents the gravitational field and the other of which exerts a force.

$$\vec{g}_1 = -G\frac{M_1}{r^2}\hat{r} \tag{7.4}$$

$$\vec{F}_2 = M_2\vec{g} \tag{7.5}$$

Of course, the second object also establishes a field which exerts and equal and opposite pull on the first object. Several useful conclusions may be established using Newton's law of gravity that will illustrate its importance in modern day scientific thinking.

7.1 Gravitational field outside of a spherical shell

The gravitational field of a uniformly dense spherical shell is the same as if the entire mass of the shell was concentrated at its geometric center. This statement can be proven by summing the contributions to the gravitational field from each element of a spherical shell of radius r and thickness t as illustrated in figure 2.3.

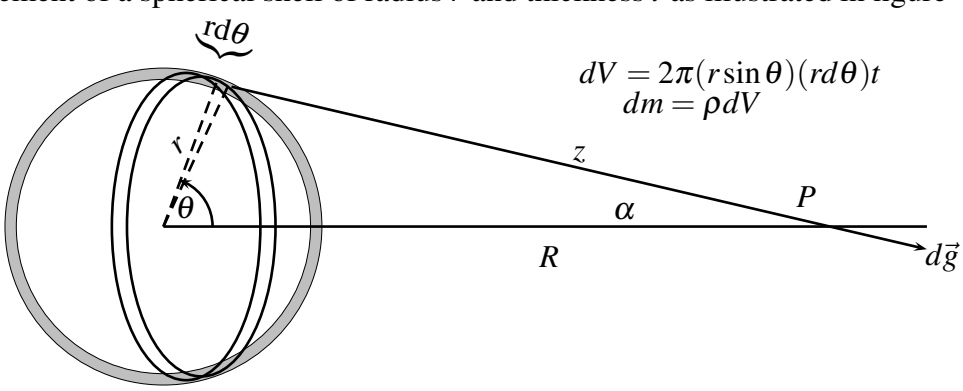


Figure 2.3: Gravitational field of a spherical shell.

To calculate the gravitational field at a point P a distance R along an axis from the center of a spherical shell we start with the field resulting from the mass

increment dm. Note that the gravitational field is a vector field making an angle α with the horizontal axis, so that the field $d\vec{g}$ due to the mass increment dm is

$$|d\vec{g}| = G\frac{2\pi\rho t r^2 \sin^2\theta d\theta}{z^2}\cos\alpha \tag{7.6}$$

$$= G\left(\frac{\pi\rho t r}{R^2}\right)\left(\frac{R^2-r^2}{z^2}+1\right)dz \tag{7.7}$$

$$\tag{7.8}$$

Now, this expression can be integrated from the limits of z at $z = R - r$ to $z = R + r$ to obtain the magnitude of the field $\vec{g}$

$$g = G\left(\frac{\pi\rho t r}{R^2}\right)\int_{R-r}^{R+r}\left(\frac{R^2-r^2}{z^2}+1\right)dz \tag{7.9}$$

$$= G\left(\frac{\pi\rho t r}{R^2}\right)\left(R^2-r^2\right)\int_{R-r}^{R+r}\frac{dz}{z^2} + G\left(\frac{\pi\rho t r}{R^2}\right)\int_{R-r}^{R+r}dz \tag{7.10}$$

$$= G\left(\frac{\pi\rho t r}{R^2}\right)\left(R^2-r^2\right)\left[\frac{1}{R-r}-\frac{1}{R+r}\right] + G\left(\frac{\pi\rho t r}{R^2}\right)[R+r-(R-r)] \tag{7.11}$$

$$= G\left(\frac{\pi\rho t r}{R^2}\right)\left(R^2-r^2\right)\left[\frac{R+r-(R-r)}{R^2-r^2}\right] + G\left(\frac{\pi\rho t r}{R^2}\right)[2r] \tag{7.12}$$

$$= G\frac{M}{R^2} \tag{7.13}$$

$$\tag{7.14}$$

The result is that the gravitational field at the point P is proportional to the total mass of the spherical shell divided by the square of the distance from the center of the shell as if the total mass were concentrated at the center of the shell.

7.2 Gravitational field outside of a dense spheroid

Since a uniformly dense spheroid is composed of many spherical shells, a useful corollary of preceding proof is that the gravitational field of a uniformly dense spheroid at a point P a distance R from its center is the same as if its entire mass were concentrated at its center.

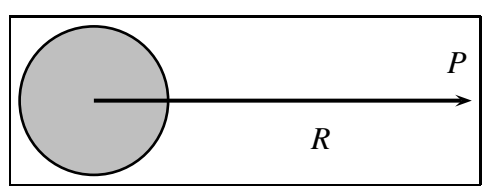


Figure 2.4: Gravitational field of a solid spheroid.

$$g = G\frac{M}{R^2} \tag{7.15}$$

7.3 Gravitational field inside a spherical shell

The gravitational field inside a uniformly dense spherical shell is zero. This can be shown by performing essentially the same integration as for a point inside the spherical shell in 2.3. In this case the point P is inside the shell and the vector R is less than the radius of the shell r.

$$
\begin{aligned}
g &= G\left(\frac{\pi\rho t r}{R^2}\right)\left(R^2-r^2\right)\int_{r+R}^{r-R}\frac{dz}{z^2}+G\left(\frac{\pi\rho t r}{R^2}\right)\int_{r+R}^{r-R}dz && (7.16)\\
&= G\left(\frac{\pi\rho t r}{R^2}\right)\left(R^2-r^2\right)\left[\frac{1}{r+R}-\frac{1}{r-R}\right]+G\left(\frac{\pi\rho t r}{R^2}\right)[r-R-(r+R)] && (7.17)\\
&= G\left(\frac{\pi\rho t r}{R^2}\right)\left(R^2-r^2\right)\left[\frac{r-R-(r+R)}{r^2-R^2}\right]+G\left(\frac{\pi\rho t r}{R^2}\right)[-2R] && (7.18)\\
&= G\left(\frac{\pi\rho t r}{R^2}\right)[2R-2R] && (7.19)\\
&= 0 && (7.20)\\
& && (7.21)
\end{aligned}
$$

For the spherical shell, the field inside the shell will be zero. Thus a second particle with mass inside the spherical shell would not experience any gravitational pull from the shell regardless of where it is located. This principle is carried over into electrostatics as will be seen in later chapters.

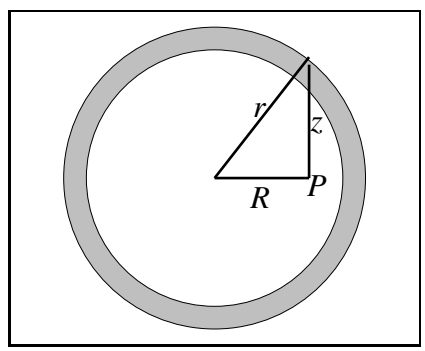


Figure 2.5: Gravitational field of a hollow shell.

7.4 Gravitational field inside solid spheroid

The gravitational field inside a uniformly dense spheroid at a point R less than the radius of the spheroid r depends only on the mass inside a spherical shell defined by the vector R and is directly proportional to R. This can be recognized by noting that none of the mass outside the shell of radius R contributes to the field at R and that the mass inside the shell contributes to the field as if it were all concentrated at the center of the spheroid.

$$g = G\frac{M'}{R^2} \tag{7.22}$$

$$= G\frac{M}{r^3}R \tag{7.23}$$

In making this calculation we assumed the earth to be a sphereoid and took the ratio of the mass inside the inner shell $M'=\frac{4}{3}\pi R^3$ to the total mass of the spheriod $M=\frac{4}{3}\pi r^3$ to obtain $M'=M\left(\frac{R}{r}\right)^3$. From this result, we see that the gravitational field of the earth decreases from a maximum at the surface of the earth toward zero at the center of the earth.

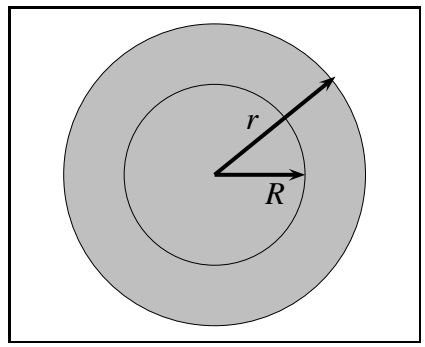


Figure 2.6: Gravitational field inside a solid spheroid.

7.5 Gravitational field at the surface of the earth

The gravitational field at the surface of a solid spheroid, such as the Earth, can easily be calculated from equation 7.15. Using the values currently published for the mean radius of the earth, $6.37100x10^6$ meters, and the mass of the earth, $5.972190x10^{24}$ kg, we obtain[5]

$$g = 9.819609 \quad meters/second^2 \tag{7.24}$$

very close to the accepted value for **standard earth gravity**, defined as precisely 9.80665 $meters/second^2$ at the 3rd General Conference on Weights and Measures in Paris in 1901 and used to define the standard weight of an object as the product of its mass and this nominal acceleration. For large planets, the variation in gravitational force with altitude can be obtained from the formula

$$\frac{dg}{g} = -2\frac{dr}{r} \tag{7.25}$$

For distances above the earth normally flown by aircraft, the variation of gravity is normally less than a few tenths of one percent of the gravitational pull at the

[5] "Solar System Exploration: Earth: Facts & Figures". NASA. 14 Feb 2011.

surface of the earth. Assuming a flight level of 12,000 meters, equation 7.25 gives a fractional variation of $2(12000/6370000) = 0.0025$ or about 0.25%.

The gravitational force or weight W of a body of mass m on or near the earth's surface can then be calculated using this value of the gravitational acceleration from

$$\vec{W} = m\vec{g} \tag{7.26}$$

The results obtained with equation 7.26 are accurate to within about 0.25% up to 1 km above the earth's surface. For this reason this equation is often used in solving physics problems.

7.6 Gravitational potential energy

The gravitational potential energy of a mass m at a distance R from a large mass M can be obtained by calculating the work done in moving the small body from infinity to a distance R from the center of the large body.

$$\triangle U = \int_{\infty}^{R} \vec{F} \cdot d\vec{r} = \int_{\infty}^{R} \frac{GMm}{r^2} dr = -\frac{GMm}{R} \tag{7.27}$$

The graph in figure 2.7 illustrates the variation of potential energy from infinity to a point near the center of gravitational force. The potential energy is negative while the kinetic energy is the difference between the potential energy and the total energy of the body. By convention, we define the gravitational potential energy of a body of mass m in the gravitational field of the earth and a distance R from the center of the earth by

$$U = -\frac{GMm}{R} \tag{7.28}$$

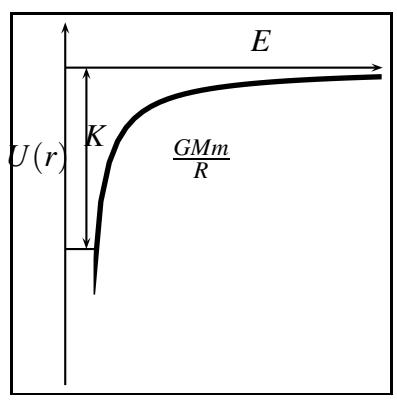


Figure 2.7: Potential energy curve.

As an example, this formula may be used to calculate the difference in the potential energies of a 1 kg body at the earth's surface and at a distance of 1000 meters above the earth's surface assuming the same values for earth's equatorial radius, mass and gravitational constant as in the preceding example.

$$\begin{aligned} U(r_e+1000) &= 6.2506078x10^7 \quad \text{Joules} \\ U(r_e) &= 6.2515878x10^7 \quad Joules \\ U(r_e+1000)-U(r_e) &= 6.2506078x10^7 - 6.2515878x10^7 = 9,800 \quad Joules \end{aligned}$$

An alternative way of calculating the potential energy of the 1 kg body would be to approximate the difference between two large numbers.

$$\begin{aligned} U &= \frac{GMm}{R_o} - \frac{GMm}{R} \\ &= \frac{GMm}{R_oR}(R-R_o) \\ &\simeq mgh \end{aligned} \tag{7.29}$$

where $RR_o \simeq R^2$ and $h = R - R_o$. Making use of this approximation the result, mgh = $(1)(9.8016)(1000)$ = 9801.6 $Joules$, will differ from the calculated value above by about 1.6 Joules.

Since the force of gravity can be written as the negative gradient of the potential energy, it is clear that **the force of gravity is a conservative force**.

$$F = -\nabla U(r) = -\hat{r}\frac{\partial}{\partial r}\left(-\frac{GM}{R}\right) = -\hat{r}\frac{GM}{R^2} \tag{7.30}$$

7.7 Escape velocity

The escape velocity of a mass m from a planet of mass M is that velocity which will allow the mass to escape the gravitational pull of the planet. This velocity can be determined by setting the potential energy at the surface of the planet equal to the kinetic energy which must be imparted to the mass m to allow escape.

$$\frac{GMm}{R} = \frac{1}{2}mV_o^2 \tag{7.31}$$

Solving for V_o gives the escape velocity.

$$V_o = \sqrt{\frac{2GM}{R}} \tag{7.32}$$

7.8 Three body system

The potential energy of a three body system is the sum of the potential energies of each pair of bodies.

$$U = \frac{GM_1M_2}{R_{12}} + \frac{GM_1M_3}{R_{13}} + \frac{GM_2M_3}{R_{23}} \tag{7.33}$$

Problems

16. Calculate the gravitational field and escape velocity at the surface of the first 6 planets in the solar system and compare to values published for our solar system in Appendix D. ans. 4.0201, 8.9232, 9.7980, 3.7994, 24.7911, 10.4444 m/sec^2; ans. 18.1, 11.5, 11.2, 15.3, 3.3, 3.6 m/sec.

17. Calculate the gravitational potential energy and gravitational force on a satellite of 500 g mass orbiting 400 miles above the surface of the earth. ans. $2.864x10^6$ Joules, 4.0419 Newtons

18. A system is composed of two thick, concentric shells. The outer and inner radii of the outermost shell are a and b respectively, while the outer and inner radii of the innermost shell are c and d respectively. The mass of the outer shell is M_1 and the mass of the inner shell is M_2. Write formulas for the gravitational field outside the system at a distance r_1 from the center, between the shells at a distance r_2 from the center and inside the innermost shell at a distance of r_3 from the center. ans. $g = G(M_1 + M_2)/r_1^2$, $g = GM_2/r_2^2$, $g = 0$

19. A spherical hollow is made inside a lead sphere of mass M, radius R and uniform density such that its circumference touches the outside of the lead sphere and the center of the lead sphere. Calculate the gravitational force on a small mass m a distance $r >> R$ from the center of the lead sphere along a line that passes through the center of the spherical hollow. ans. g=$\frac{GMm}{r^2}\left(1 - 1/8\left(1 - \frac{R}{2r}\right)^2\right)$

8 Elastic forces

8.1 Hooke's law

Another conservative force that often arises in nature, and one which has played a major role in the development of physics, is that of the elastic system. The first example of an elastic system that we will study consists of a mechanical spring attached to a rigid support and stretched in a horizontal direction. The unique characteristic of the spring is that the force necessary to stretch the spring, and the force which a stretched spring exerts on another object, is directly proportional to the distance the spring has been stretched.

$$F = kx \tag{8.1}$$

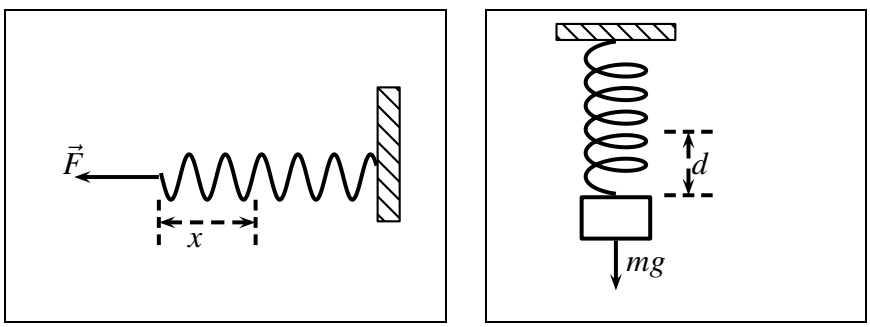


Figure 2.8: Stretching a spring with a Force F (A) and by the force of gravity on a mass m (B).

This law is known as **Hooke's law**, named after the 17th century British physicist Robert Hooke who published his work in 1678. The constant k is known as the **spring constant** and has units $\frac{Newtons}{meter}$. The potential energy stored in the spring as a result of stretching can be calculated by integration of equation 8.1 over the distance the spring is stretched. This potential energy may be considered energy that is stored in the spring.

$$U = \int_0^x \vec{F} \cdot \vec{x} dx = \int_0^x kx dx = \frac{1}{2}kx^2 \tag{8.2}$$

If a block of mass m is suspended by the spring, the spring will stretch a distance d until an equilibrium is reach between the upward force of the spring kd and downward force of gravity mg.

$$mg = kd \tag{8.3}$$

This equilibrium provides a convenient method for measuring the spring constant in the laboratory from the weight of the block and distance the spring is stretched.

$$k = \frac{mg}{d} \tag{8.4}$$

Two springs may also be used in either parallel or series combination, which results in modification of the spring constants. In the case of a parallel combination, the effective spring constant may be calculated by noting that the total weight of the mass m will be supported by two springs stretched by the same amount d.

$$\begin{aligned} mg &= k_1 x + k_2 x && (8.5)\\ &= (k_1 + k_2)x \quad \text{so that} && (8.6)\\ k_{eff} &= k_1 + k_2 && (8.7) \end{aligned}$$

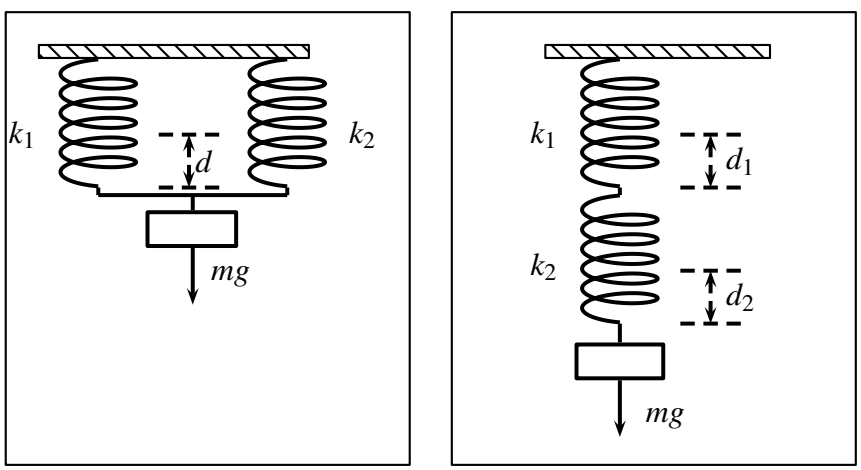


Figure 2.9: Springs in parallel and series.

In the case of a series combination, the first spring is stretched an amount x_1

and the second spring is stretched by x_2 so that $x = x_1 + x_2$.

$$mg = k_1 x_1 + k_2 x_2 \quad \text{so that} \tag{8.8}$$

$$x = x_1 + x_2 = \left(\frac{mg}{k_1} + \frac{mg}{k_2}\right) \tag{8.9}$$

$$= \frac{k_2 + k_1}{k_1 k_2} mg = \frac{mg}{k_{eff}} \quad \text{so that} \tag{8.10}$$

$$k_{eff} = \frac{k_1 k_2}{k_1 + k_2} \tag{8.11}$$

The force exerted by a spring as discussed above can be considered linear in that the force is always proportional to the displacement. While this principle holds true over limited ranges, forces in nature that are proportional to displacement often depend on proportionality constants that vary with elongation and are therefore non-linear. The constant of proportionality may depend on temperature, internal properties of the spring and many other parameters. Hook's law will also arise in similar form in the stretching, compressing, and twisting of physical objects that will be discussed throughout this text.

8.2 Total energy and conservation in elastic systems

It is instructive to examine the potential, kinetic and total energies of a simple pendulum which is initially hung from a support and allowed to reach an equilibrium position at (B) balancing the pull of gravity against that of the spring; then pulled down a distance s to a point (A) and released. At the lowest position (A), the kinetic energy of the mass will be zero while the potential energy will be $(U_o) = \frac{1}{2}ks^2$. The total energy of the mass will also be $(E_o) = \frac{1}{2}ks^2$. After release the mass will pass through the equilibrium position (B) where its potential energy will be zero. At position (B) the kinetic energy $K_o = \frac{1}{2}mV^2$ of the mass will equal the

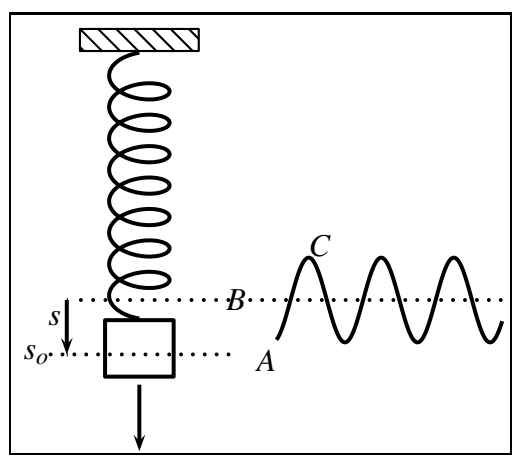


Figure 2.10: Spring pendulum.

total energy $(K_o) = E_o = \frac{1}{2}ks^2$, which allows the velocity to be calculated from

$$V = \sqrt{\frac{ks}{m}}. \tag{8.12}$$

After passing through the equilibrium point the mass will continue to the top of its motion where its kinetic energy will again be zero and its potential and total energies will be the same as at the starting point $(U_o) = E_o = \frac{1}{2}ks^2$. Then the motion of the mass will retrace its path.

8.3 Oscillatory systems

At any point in its path, the acceleration of the mass attached to the spring in figure 2.10 is determined by the force of the spring which allows an equation of motion for the spring mass to be written as

$$ma = -ks \quad \text{so that} \tag{8.13}$$

$$m\frac{d^2s}{dt^2} + ks = 0 \quad \text{or} \tag{8.14}$$

$$\frac{d^2s}{dt^2} + \omega^2 s = 0 \tag{8.15}$$

The constant $\omega = \sqrt{\frac{k}{m}}$ has units of inverse seconds $[\omega] = sec^{-1}$; and the solution to this differential equation is

$$s = A\cos\omega t + B\sin\omega t \tag{8.16}$$

The constants, A and B are determined by the initial conditions for the spring at the time of its release, $s = s_o$ and $t = 0$, from which we find $B = 0$ so that the equation of motion for the spring becomes

$$s = s_o\cos\omega t \tag{8.17}$$

The constant ω can be related to the period of the motion by noting that $s = s_o$ when $t = T$ which gives

$$T = \frac{2\pi}{\omega} \tag{8.18}$$

The velocity and acceleration can be determined by differentiating equation 8.17

$$s = s_o\cos\omega t \tag{8.19}$$

$$\dot{s} = -\omega s_o\sin\omega t \tag{8.20}$$

$$\ddot{s} = -\omega^2 s_o\cos\omega t = -\omega^2 s \tag{8.21}$$

That these equations are compatible with the foregoing equation 8.1 may be checked by computing $F = m\ddot{s} = m\omega^2 s_o \cos\omega t = m\omega^2 s = ks$.

The spring force is derivable from a single valued function $U = \frac{1}{2}ks^2$ and is therefore a conservative force. We can define this single valued function of position as the potential energy and then compute the kinetic and total energies of the mass as follows

$$U = \frac{1}{2}ks^2 \tag{8.22}$$

$$K = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}m\omega^2 s_o^2 \sin^2\omega t = \frac{1}{2}k\left(s_o^2 - s^2\right) \tag{8.23}$$

$$E = K + U = \frac{1}{2}ks_o^2 \tag{8.24}$$

Problems

20. A 100 g mass under the influence of gravity alone is found to stretch a spring 5 cm when attached to it. What is the force constant of the spring? ans. 19.62 N/m

21. What is the effective constant of two springs, one having a spring constant 5 N/m and the other a spring consant 10 N/m, when attached in series? ans. 3.33 N/m

22. What would be the effective constant if the two springs were in parallel? ans. 15 N/m

23. A mass of 1 kg is attached to a spring with a constant of 100 N/m, pulled down 5 cm and released. What is the period and frequency of oscillation? ans. 0.628 seconds, 1.59 Hz

24. Suppose that the earth had a uniform density and that a hypothetical hole could be drilled through the center of the earth to the other side. Also suppose that inside the hole there is a perfect vacuum. If a particle of mass m were dropped into the hole it would execute oscillatory motion. Calculate the period of oscillation for the particle. ans. $T = 84$ minutes

9 Force of friction

When one block slides over the surface of another object the roughness of the surface impedes that motion. We can think of the force of friction as arising from the meshing of irregularities in the surface of the two blocks which tends to resist movement of the block.

There are two types of friction called **static friction** and **kinetic friction**. The force of static friction always adjusts itself to match the applied force as long as no motion occurs. When the irregularities in the surfaces begin to slide over one another, the force of friction changes to kinetic friction and always adjusts itself to its maximum value somewhat as illustrated in figure 2.11

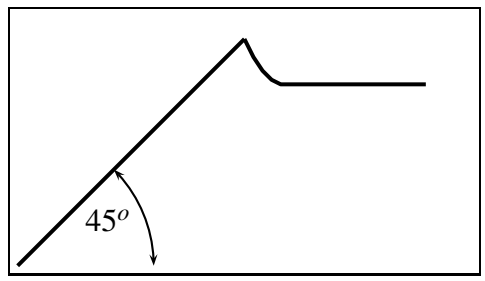


Figure 2.11: Variation of frictional resistance with applied force.

This retardation effect can be approximated as a force of friction $\vec{f}$ acting in a direction opposite to the motion of the block. The magnitude of the frictional force is proportional to the normal force pushing the block against the surface

$$f = \mu N \tag{9.1}$$

where μ is known as the **coefficient of friction** and represented by μ_s in the case of static friction and μ_k in the case of kinetic friction. In figure 2.12 a force F pulls a block of mass m along a flat surface. The force of gravity pulling the block downward normal to the surface is mg and the force of friction retarding motion of the block is $f = \mu mg$. The resulting unbalanced force $F - f$ produces an acceleration a in the direction of motion, which may be calculated by

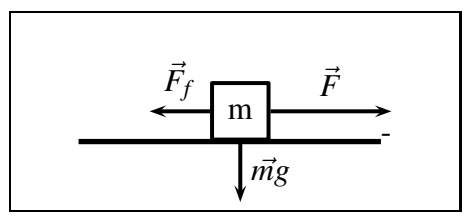


Figure 2.12: Pulling a block along a horizontal plane.

$$f - \mu mg = ma \tag{9.2}$$

$$a = \frac{F - \mu mg}{m} \tag{9.3}$$

For example, suppose the mass of the block is 2 kg so that the weight of the block is 19.62 N, while the maximum value of static friction is $\mu_s = 0.7$ and $\mu_k = 0.4$. If a force F of 10 N is applied to the block, the force of friction $0.7 \times 19.6 = 13.7$ N will overcome the applied force and no movement will occur. If a force F of 15 N is applied, the block will start moving, the force of friction will drop to $0.4 \times 19.6 = 7.85$ N and the acceleration will be $(15.0 - 7.85)/2 = 3.58$ m/sec^2.

If the applied force $\vec{F}$ is inclined at an angle θ with the horizontal, as illustrated in figure 2.13, the applied force must be resolved into components parallel to and normal to the surface. In this case the component of the applied force normal to the surface reduces the normal force pressing the block against the surface and thus the force of friction. As a result, the block will slide faster.

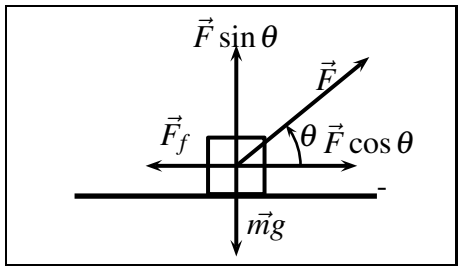


Figure 2.13: Inclined force pulling a block along a horizontal plane.

$$f = \mu\,(mg - F\sin\theta) \tag{9.4}$$

$$F\cos\theta - \mu\,(mg - F\sin\theta) = ma \tag{9.5}$$

$$a = \frac{F\cos\theta - \mu\,(mg - F\sin\theta)}{m} \tag{9.6}$$

On an inclined plane the normal force pushing the block against the surface would be increased as the angle of inclination θ increases. As illustrated in fig 2.14, where the applied force F is horizontal, the normal weight of the block pushing against the surface is $\mu mg\cos\theta + F\sin\theta$.

A horizontal force pulling a block up an inclined plane may be resolved into two components, one parallel to the incline and the other normal to the incline. The normal component of the applied force will increase the force with which the block is

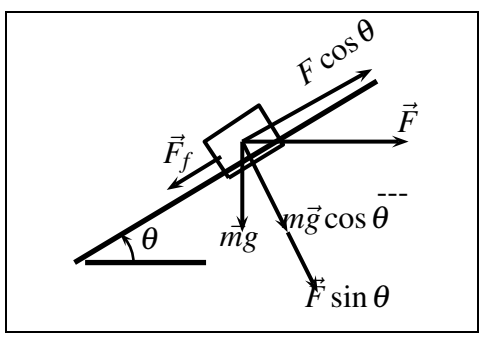


Figure 2.14: Horizontal force pulling block along inclined plane.

pressed against the incline and therefore the frictional force. If the coefficient of friction, or the angle θ, is great enough a point may be reached as the applied force $\vec{F}$ is increased that the force of friction will counterbalance the effect of the applied force in pulling the block up the incline. At this point motion will stop.

$$f = \mu\left(mg\cos\theta + F\sin\theta\right) \tag{9.7}$$

$$F\cos\theta - \mu\left(mg\cos\theta + F\sin\theta\right) = ma \tag{9.8}$$

$$a = \frac{1}{m}\left(F\cos\theta - \mu\left(mg\cos\theta + F\sin\theta\right)\right) \tag{9.9}$$

Problems

25. A 10.0 kg block is placed on a table where the maximum coefficient of static friction is 0.600 and the coefficient of kinetic friction is 0.400. A string is attached to the block and passed over a massless, frictionless pulley and attached to a 5.00 kg block. Will the 10.0 kg block slide across the table? What is the mass of the block that must replace the 5.00 kg block to cause motion? If the 5.00 kg block is replaced with a 10.0 kg block, what will be the acceleration of the system? What will be velocity of the system after 2.00 seconds? ans. No, $\geq$ 6.00 kg, 2.94 m/sec^2, 5.89 m/sec

26. Suppose that a 10.0 kg block is placed on a board inclined at 37.0 degrees with the horizontal and that the maximum coefficient of static friction is 0.60 while the coefficient of kinetic friction is 0.40. Will the 10.0 kg block slide down the incline? What will its acceleration be? Yes, 1.21 m/sec^2

27. A dock worker attempts to push a 10.0 kg block up a board inclined at 37.0 degrees with the horizontal by applying a force of 100 Newtons parallel to the horizontal. Take the maximum coefficient of static friction is 0.600 and the coefficient of kinetic friction is 0.400. Will the 10.0 kg block move, and if not what is the maximum force necessary? No, 242 N

28. Assume a block with a mass of 2.00 slugs is placed on a table where the coefficient of kinetic friction is 0.400. A string is attached to the block and passed over a massless, frictionless pulley to another mass of 3.00 slugs. Will the system move and what will be its acceleration? What will be its velocity after falling 3.00 seconds? How far will the system have moved in that time? ans. 14.2 ft/sec^2, 42.5 ft/sec, 63.7 ft.

29. A 2 kg block rests on a plane inclined at 30 degrees with a coefficient of kinetic friction of 0.400. A horizontal force of 20 N pulls the block up the incline. What will be the acceleration of the block up the incline? ans. 1.70 m/sec^2

10 Kinematics

10.1 Definitions and terminology

The purpose of kinematics is to provide a means of locating an object with respect to an established reference point and for calculating the velocity and acceleration of the object. For this purpose, we will utilize the Cartesian coordinate system and define the position of an object with the three cartesian coordinates (x, y, z) as illustrated in figure 2.15.

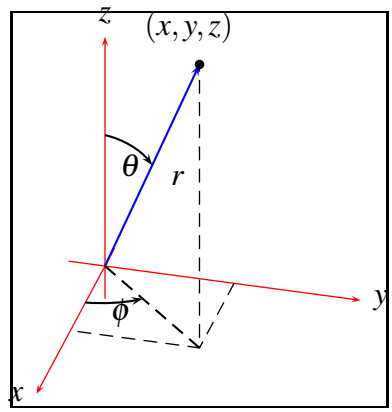


Figure 2.15: Particle located in three-dimensions

In this coordinate system the vector $\vec{r}$ defines the distance from the origin to the particle, the angle θ measures the inclination of $\vec{r}$ with respect to the z-axis and the angle ϕ measures the angle from the positive xz-plane to the point. In the Cartesian coordinate system the position of the particle can be represented in vector form as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{10.1}$$

where the coordinates (x, y, z) are related to the vector $\vec{r}$ and angles θ and ϕ by

$$x = r\sin\theta\cos\phi \tag{10.2}$$
$$y = r\cos\theta\cos\phi \tag{10.3}$$
$$z = r\cos\theta \tag{10.4}$$

The velocity of the particle $\vec{V}$ is defined as the time rate of change of the position vector $\vec{r}$ and is tangent to the surface on which the particle is moving but not necessarily perpendicular to $\vec{r}$.

$$\vec{V} = \hat{i}V_x + \hat{j}V_y + \hat{k}V_z \tag{10.5}$$

where the components of the velocity are defined by

$$V_x = \frac{dx}{dt} \tag{10.6}$$

$$V_y = \frac{dy}{dt} \tag{10.7}$$

$$V_z = \frac{dz}{dt}. \tag{10.8}$$

The acceleration of the particle $\vec{a}$ is defined as the first derivative of the velocity vector $\vec{V}$ or the second derivative of the position vector $\vec{r}$.

$$\vec{a} = \hat{i}a_x + \hat{j}a_y + \hat{k}a_z \tag{10.9}$$

where the components of the acceleration are defined by

$$a_x = \frac{dV_x}{dt} \tag{10.10}$$

$$a_y = \frac{dV_y}{dt} \tag{10.11}$$

$$a_z = \frac{dV_z}{dt} \tag{10.12}$$

10.2 Kinematic equations

In the case of constant acceleration, the defining equation can easily be integrated to obtain an expression for any component of the velocity at any point in time. Representing that component with the letter s

$$\int_{V_{os}}^{V_s} dV_s = \int_0^t a_s dt \tag{10.13}$$

$$V_s(t) = V_{os} + a_s t \tag{10.14}$$

Similarly, the defining equation for velocity can be integrated to obtain an expression for any component of the distance traveled.

$$\int_{s_o}^{s} ds = \int_0^t V_s dt \tag{10.15}$$

$$= V_{os}\int_0^t dt + a_s \int_0^t t dt \tag{10.16}$$

$$s = s_o + V_{os}t + \frac{1}{2}a_s t^2 \tag{10.17}$$

Solving equation 10.14 for t and substituting into equation 10.17 results in a solution for the velocity in terms of the distance traveled.

$$V_s^2 = V_{os}^2 + 2a_s(s - s_o) \tag{10.18}$$

10.3 Application to vertical motion

Consider the case of a projectile of mass 50 grams fired upward with an initial velocity $V_{oy} = 10$ m/sec from a height of 5 meters above ground. The only force acting is that of gravity pulling downward with an acceleration of $a = 9.81\ m/sec^2$. All other components of velocity and acceleration are zero. Substituting $V_y = 0$ into equation 10.14 will reveal that the projectile travels upward for 1.02 seconds before the pull of gravity stops the upward motion.

$$0 = 10 + 9.81t \tag{10.19}$$

$$t = \frac{10}{9.81} = 1.02\ sec \tag{10.20}$$

Substituting $V_y = 0$ into equation 10.18 will reveal that the projectile reaches a height above ground of 10.10 meters.

$$0 = (10)^2 + 2(9.8)(y - 5) \tag{10.21}$$

$$y = 5 + \frac{100}{(2)(9.81)} = 10.1\ meters \tag{10.22}$$

At the highest point of the trajectory, the kinetic energy will be zero but the potential energy of the bullet is obtained by substituting the mass of the bullet into the equation 7.29, $U = mgh$, to obtain $U = 4.95$ Joules.

$$U = (0.050)(9.81)(10.1) = 4.95\ Joules \tag{10.23}$$

This energy will be converted to kinetic energy when the bullet reaches the ground. Using the formula $\frac{1}{2}mV^2$ set equal to the potential energy at the top of the trajectory shows that the velocity of the mass will be 14.07 m/sec when it reaches ground.

$$4.95 = \frac{1}{2}(0.05)V_y^2 \tag{10.24}$$

$$V_y = \sqrt{\frac{(2)(4.95)}{0.05}} = 14.07\ meters/second \tag{10.25}$$

This velocity is greater that the initial velocity with which the mass was fired upward because of the additional 5 meters the projectile fell before it reached ground. This result can be checked by calculating the velocity on impact with the ground from equation 10.18 using $V_o = 0$, $s = 10.10$ meters and $a = 9.81$ meters/second to obtain the same result.

$$V_y = 2(9.81)(10.1) \tag{10.26}$$

$$V_y = \sqrt{2(9.81)(10.1)} = 14.07\, meters/second \tag{10.27}$$

The kinematic equations always afford a means of checking the result obtained using the equations in one sequence by using the equations in another sequence.

10.4 Application to motion in a plane

Another useful application of kinematics is calculation of the trajectory of a bullet fired with an initial velocity V_o at an angle θ with respect to the horizontal from a height y_o above a level plain. In this case the information available for calculating the trajectory is the variables (V_o, θ, y_o) plus the acceleration of gravity, known to be $g = 9.81\ m/sec^2$ and directed downward in the negative y direction. The initial velocity V_o can be resolved into x and y components, but there is no accelerating force in the x direction. We will assume that there is no atmospheric friction to alter the trajectory of the projectile.

Using these parameters, the equations relating the coordinates y and x can be obtained by substituting into the kinematic equations.

$$a_x = 0 \tag{10.28}$$

$$V_x = V_o \cos\theta \tag{10.29}$$

$$x = x_o + V_o t \cos\theta \tag{10.30}$$

$$V_x^2 = V_o^2 \cos^2\theta \tag{10.31}$$

$$a_y = -9.81 \tag{10.32}$$

$$V_y = V_o \sin\theta - gt \tag{10.33}$$

$$y = y_o + V_o t \sin\theta - \frac{1}{2} g t^2 \tag{10.34}$$

$$V_y^2 = V_o^2 \sin^2\theta - 2g\,(y - y_o) \tag{10.35}$$

Still another equation relating the coordinates x and y directly can be obtained by solving for the time t in the equation for V_x and substituting into the equation for

y. This results in

$$y = y_o + (x - x_o)\tan\theta + \frac{g}{2V_o^2\cos^2\theta}(x - x_o)^2 \tag{10.36}$$

which makes it clear that under the ideal circumstances, the trajectory of the projectile will be a parabola. An equation for the range, $R = x - x_o$, of the projectile can be obtained from this equation by setting y=0 to get

$$\left(\frac{g}{2V_o^2\cos^2\theta}\right)(x - x_o)^2 - (x - x_o)\tan\theta - y_o = 0 \tag{10.37}$$

and solving the resulting quadratic expression for the range.

$$R = \left(\frac{V_o\cos\theta}{g}\right)\left[V_o\sin\theta + \sqrt{V_o^2\sin^2\theta + 2gy_o}\right] \tag{10.38}$$

The angle of inclination which will give the maximum range can be found by differentiating equation 10.38 and setting the result equal to zero to obtain

$$\theta = \arccos\sqrt{\frac{2gy_o + V_o^2}{2gy_o + 2V_o^2}} \tag{10.39}$$

When the initial height is zero, the angle $\theta = \arccos\sqrt{\frac{1}{2}} = 45^o$. When the initial height is not zero, the angle for maximum range will be less than 45°, and when the projectile is fired from below ground the angle for maximum range will be greater than 45°.

For example, suppose a baseball is hit with a bat at a height of 4.00 feet and given an initial velocity of 100 feet/second at an angle of 30.0 degrees. Substituting this data into the above equations and using the acceleration of gravity 32.174 $meters/second^2$, the following equations of motion are obtained for the baseball.

$$a_x = 0 \tag{10.40}$$

$$a_y = -32.174\ ft/sec^2 \tag{10.41}$$

$$V_{ox} = 86.6\ ft/sec \tag{10.42}$$

$$V_{oy} = 50.0\ ft/sec \tag{10.43}$$

$$V_x = 86.6t \tag{10.44}$$

$$V_y = 50.0 - 32.174t \tag{10.45}$$

$$y = 4 + 50.0t - \frac{1}{2}(32.174)t^2 \tag{10.46}$$

$$V_y^2 = 2500 - 64.348(y - 4) \tag{10.47}$$

Solving simultaneously by substitution we obtain the following results.

$$t(maxheight) = 1.55\ seconds \tag{10.48}$$
$$y_{max} = 42.85\ feet \tag{10.49}$$
$$x(maxheight) = 134.58\ feet \tag{10.50}$$
$$t(range) = 3.186\ seconds \tag{10.51}$$
$$Range = 275.9\ feet \tag{10.52}$$

11 Motion with Air Friction

The formulas of the last section enable us to calculate trajectories of projectiles in the ideal case of no air friction. This, however, was found to be unacceptable in military applications of artillery bombardment, launching of rockets and other projectiles. For this reason, the preceding formulas must be amended for practical applications. As a projectile moves through air its motion is opposed by a force directed opposite to the velocity and commonly known as the "drag force".

$$F_D(V) = \frac{1}{2}\rho A V^2 C_D \tag{11.1}$$

In this formula, ρ is the density of air, approximately 01.2 kg/m^3, A is the cross-sectional area of the projectile, V is the velocity and C_D is the unitless coefficient of drag. The coefficient of drag is a function of velocity as well as the geometrical shape of the projectile and the roughness of its surface. This drag force produces an acceleration in the direction opposite to motion (or de-acceleration) $a_D = F_D/m$. This de-acceleration will be larger the smaller the mass, thus affecting small projectiles such as bullets much more than large projectiles such as artillery shells. Since the drag force is directed opposite to the velocity vector, the equations of motion must be amended as follows.

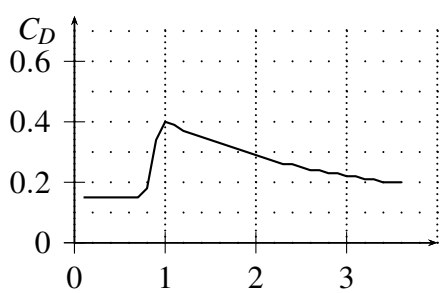


Figure 2.16: Drag coefficient for projectile in air versus Mach number.

$$V_x = V_{ox} - a_{Dx}t \tag{11.2}$$

$$V_y = V_{oy} - \frac{1}{2}(g + a_{Dy})t^2 \tag{11.3}$$

In general, the drag coefficient depends on several parameters including the geometrical shape of the projectile, the surface roughness, the temperature and humidity of the air and most important the velocity of the projectile. The drag coefficient undergoes an increase near the speed of sound in air rising from a nominal value of about 0.1 to about 0.5 and decreasing at speeds above the speed of sound. This variation is depicted in figure 2.16 and generally holds true for projectiles ranging from small caliber rifle bullets up to large naval guns.

11.1 Trajectory of a ball with air friction

With these modifications, the equation of motion for a projectile cannot be solved analytically, but it can be solved by computation with the aid of a spreadsheet. A hypothetical example of the trajectories of a baseball for the example above with and without air resistance is provided in figure 2.17 to illustrate the effect of air resistance. For this example, the mass of the ball was taken as 0.010 slugs, the radius as 0.120 ft, and the coefficient of drag as 0.27. It is assumed that the ball is driven upward with a velocity of 100 ft/sec at an angle of 30.0 degrees with the horizon. As illustrated, the trajectory of the ball without air resistance reaches a height of 42.85 feet and the ball travels a distance of 276 feet before returning to the ground. In the case of air resistance, the trajectory of the ball is shortened to 220 feet while the maximum height reached is 38.6 feet.

11.2 Trajectory of a cannon ball with air friction

Another interesting example is provided by calculating the data and range for a 1200 kg projectile fired at an angle of 45 degrees relative to the horizon with a velocity of 762 m/sec from a 16-inch naval gun. The radius of the projectile will be taken as 0.203 m and the density of air 1.2 kg/m^3. For the purpose of calculating the coefficient of drag, a straight line fitted to the data for C_D above Mach 1 will be used to calculate the drag at a given velocity. Using the computational method the results illustrated in figure 2.18 were obtained. Although this approach is only an approximate method, the result of 39 km for the range of a shell from a 16-inch naval gun closely matches those obtained in actual use of the gun.

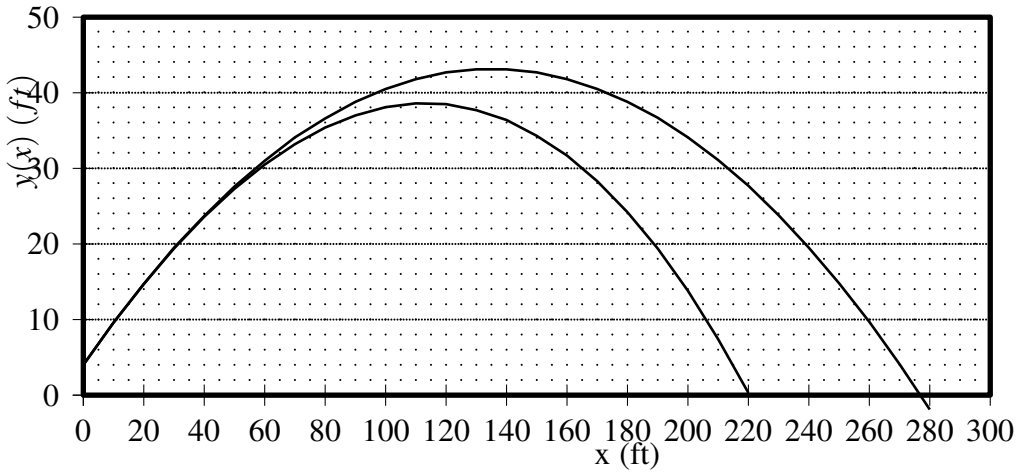


Figure 2.17: Trajectory of baseball with and without air resistance.

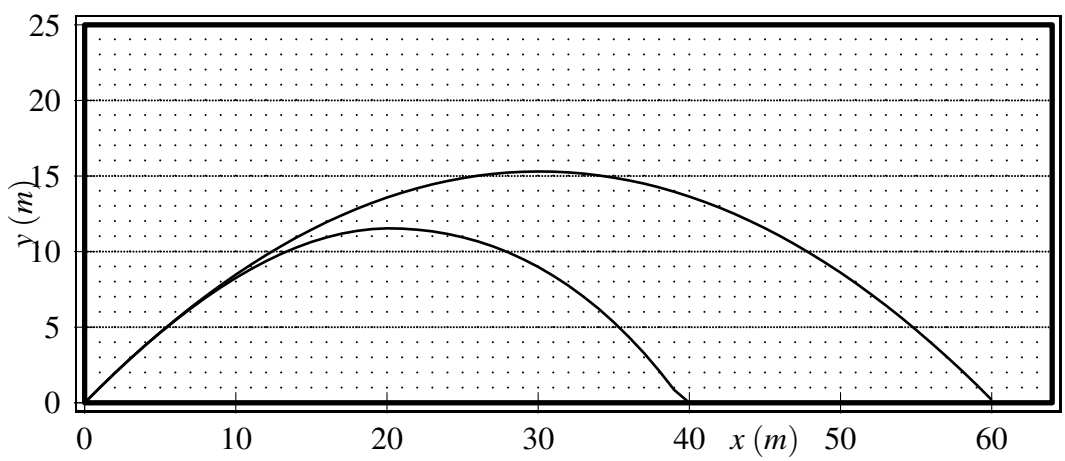


Figure 2.18: Trajectory of shell from a 16-inch naval gun with and without air friction.

11.3 Terminal velocity

When an object falls through the air toward the earth, the force of gravity mg pulls it downward while the force due to air drag $\frac{1}{2}\rho AV^2C_D$ pulls against the force of gravity. When the effect of each force counterbalances the other, the velocity of the object remains constant and is called the **terminal velocity**. An expression for the terminal velocity can be obtained by equating these two forces and solving for

the velocity.

$$V = \sqrt{\frac{2mg}{\rho A C_D}} \tag{11.4}$$

This calculation ignores the buoyant force of the air on the falling object. If the medium through which the object falls is sufficiently dense that the buoyant force must be taken into account, a force equal to the weight of the medium displaced by the object must be added to the force of air drag.

Problems

30. Calculate the range for a 155 mm shell fired from a $M114$ howitzer with a muzzle velocity of 563 m/sec at an angle of 45-degrees without and with air friction. For this calculation use straight line fit to the plotted drag coefficients, take the density of air to be 1.24 kg/m^3, the weight of the 155 shell to be 43.89 kg and the diameter of the projectile to be 0.154 m. Solve for case with air friction using spreadsheet. ans. 32.3 km w/o air resistance, 15.7 km with air resistance.

31. Calculate the drop in inches for a .308 caliber round nose rifle bullet at 500 yards (457 meters) fired with a muzzle velocity of 858.6 m/sec at an angle of 0-degrees. Also calculate the velocity, energy and momentum of the bullet. For this calculation take the density of air to be 1.24 kg/m^3, the weight of the bullet to be 150 grains (9.72 g) and the diameter of the bullet to be 0.308 in (0.782 cm). Obtain the drag coefficient from figure 2.16 and make the necessary conversions between metric and British units. Use spreadsheet to calculate trajectory. ans. drop = 1.25 m; velocity = 855.5 m/sec; energy = 3557 Joules and momentum = 8.32 kg-m/sec.

32. What is the terminal velocity in free fall for a baseball, golfball, hailstone and raindrop with the following parameters? ans. 33.0, 32.4, 13.9 and 9.30 m/sec.

object	radius (cm)	mass (gm)	C_D
baseball	3.66	145	0.500
golfball	2.10	460	0.500
hailstone	0.500	0.480	0.500
raindrop	0.200	0.034	0.500

33. Derive an expression for the terminal velocity of a spherical object in free fall taking account of the buoyant force of the medium. For this derivation, let the density of the medium be ρ_m and that of the object be ρ. ans. $V = \sqrt{\frac{4gd}{3C_D}\frac{\rho-\rho_m}{\rho_m}}$

34. Compare the trajectories of a baseball of mass 0.00976 slugs and radius 0.121 feet with that of a softball of mass 0.122 slugs and radius 0.159 feet. Assume each ball is struck 4.00 feet above home plate and given an intial velocity of 100 ft/sec at an angle of 30 degrees with the horizontal and that the coefficient of drag equal to 0.270 and plot their trajectories. Use spreadsheet to calculate trajectories. ans. Range baseball = 218.5 ft; softball = 266.5 ft. Why is the range of the softball greater than the range of the baseball.

12 Impulse and collision force

Studies thus far have been limited to continuous forces. However an important field of mechanics involves estimating the force of collisions. This can be accomplished by defining the **Impulse** as

$$\vec{I} = \int_{t_1}^{t_2} \vec{F} dt \tag{12.1}$$

In this definition, we are assuming that the force acts only for a short duration such as that depicted in figure 2.19 which implies, using the definition of force as dp/dt, that the Impulse is the change of momentum in the collision and carries the same units $kg - m/sec$. The impulse force can be calculated in several ways. Consider, for example, a brick of mass 2.00 kg falling from a third story window 10.0 meters to the sidewalk below. Neglecting drag from air friction, the velocity

of the brick on impact will be

$$v = \sqrt{2(9.81)(10)} = 14.0 \quad m/sec \tag{12.2}$$

giving it a kinetic energy and momentum of

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(2.00)(14.0)^2 = 98.1 \quad \text{Joules} \tag{12.3}$$

$$p = mv = (2.00)(14.0) = 28.0 \quad kg - m/sec \tag{12.4}$$

Two cases can now be considered, one for a soft collision and one for a hard collision. If the brick penetrates 10.0 cm into the earth before stopping and remains intact, we can assume it took 0.100/14.0 = 0.007 seconds to come to a stop. Using the definition above, the average impulse force can then be calculated as

$$F_{av} = \frac{\Delta p}{\delta t} = \frac{28.0}{0.007} = 4,000 \quad \text{N.} \tag{12.5}$$

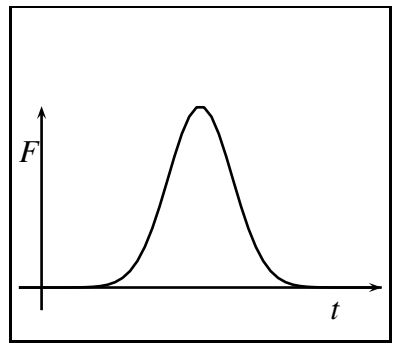


Figure 2.19: Impulse Force.

The second case is for the brick striking a hard sidewalk that does not yield to the collision and the brick remains intact. In this case we can estimate the time of impact by doubling the time for a sound wave traveling about 384 m/sec to travel twice the length of the brick, about 10.0 cm, to obtain a collision time of 0.100/384 = 0.0005 seconds. This will result in a much greater average impulse force of

$$F_{av} = \frac{\Delta p}{\delta t} = \frac{28.0}{0.0005} = 56,000 \quad \text{N.} \tag{12.6}$$

As another example, consider the impact of a bird weighing 1 pound and having a mass of 0.03 slugs with an airplane flying at 600 mph or 880 ft/sec. In this case we can assume the time for collision is the time for the airplane to overrun the bird which may be 1 foot long or about 1/880 = 0.001 seconds. The bird's momentum relative to the airplane can be taken as mv = (0.03)(880) = 26.4 lb-sec. Then taking the average force as the rate of change of momentum

$$F = \frac{26.4}{0.001} = 26,400 \quad \text{lb} \tag{12.7}$$

or about 12 tons.

35. What is the average force of impact of a .45 caliber bullet of mass 240 grains (0.001065918 slugs) traveling 500 ft/sec upon hitting a block of soft ballistic gel and traveling 6.00 inches before stopping? ans. 266.5 lb If the block of ballistic gel has the mass of a normal man, 192 lb, what will be the backward velocity of the block after collision? ans. 0.0893 ft/sec

36. What is the force of impact of a .30 caliber bullet of mass 100 grains (0.000444 slugs) and 0.5 inches long traveling at 2000 ft/sec upon hitting a hard metal surface? ans. 11,984 lb

37. The famous Parrot gun used in the civil war fired a round 10 lb projectile 6 inches long with an initial muzzle velocity of 1,111 ft/sec. The projectile had a velocity of about 700 ft/sec at 3500 yards. What would the force of impact be for a 10 lb projectile hitting a hard surface. ans. 245,045 lb

38. What is the force of impact of an egg weighing 2 ounces and being spherical with diameter of 2.00 in after falling a distance of 64.0 feet from rest as it strikes a man on the head. Assume the man's head resists the impact and does not move. ans. 95.9 lb

13 Rocket equation

As an illustration of the use of equation 6.4 relating force and momentum, a heuristic derivation of the **rocket equation** will be given. More detailed derivations can be found in advanced engineering texts.[6] Taking the mass of a rocket as M_R and its velocity at any point in its trajectory as V_R and noting that in the absence of external forces, the time derivative of momentum is zero allows us to write

$$F = M_R \frac{dV_R}{dt} + V_R \frac{dM_R}{dt} = 0. \tag{13.1}$$

Setting the initial mass of the rocket to be M_o and assuming a constant rate of burn for the fuel $R = \frac{dM_f}{dt}$, we see that

$$M_R = M_o - Rt. \tag{13.2}$$

Also, we can replace the mass of the rocket with the mass of the payload and mass of fuel, $M_R = M_{PL} + M_F$, and note that $dM_R = dM_F$. The exhaust velocity

[6] J.E.A. John and W. L. Haberman, "Introduction to FLUID MECHANICS, Prentice-Hall, Inc., New Jersey, 2nd edition, (1980) page 102

of the burning fuel c must be added to the velocity of the rocket in the second term of equation 13.1 with the result that the velocity of the rocket V_R is replaced by the exhaust velocity of the fuel relative to the inertial frame of reference or $V_e = c - V_R$. As a result

$$(M_o - Rt)\frac{dV_R}{dt} = RV_e \tag{13.3}$$

Upon integrating, we obtain the so called "rocket equation"

$$V_R = -V_e \ln\left(1 - \frac{Rt}{M_o}\right) \tag{13.4}$$

The thrust provided by the rocket is also defined as

$$T = RV_e \tag{13.5}$$

As an example, consider the Saturn 1B rocket which has a sea level thrust of 7.295×10^6 N, consumes fuel at a rate of about 2832 kg/sec, has an initial mass of 586,000 kg and a fuel load of 400,000 kg. Using the above equations, the exhaust velocity, time of burn and final rocket velocity is found to be:

$$V_e = \frac{7.295 \times 10^6}{2832} = 2576\, m/sec \tag{13.6}$$

$$t = \frac{400,000}{2832} = 141\ seconds \tag{13.7}$$

$$V_R = -2576 \ln\left(1 - \frac{400,000}{586,000}\right) = 2956\, m/sec \tag{13.8}$$

At this point, the first stage would separate and the second stage would carry the rocket into space.

39. A rocket similar to one used to carry a AIM-9 sidewinder missile carries 30.0 kg of fuel and provides 17,800 N thrust and burns fuel at the rate of 10.0 kg/sec. The missile and payload weighs 70.0 kg. What velocity can be achieved? ans. 996 m/sec

Chapter 3

ROTATIONAL MOTION

In this chapter, the concepts of linear dynamics and non-rotational systems will be extended to rotational systems. Completion of this chapter will provide the formulas and principles to address problems in any mechanical system.

13 Rotational Dynamics

Our purpose in this section is to define the fundamental parameters used to describe rotational motion and to learn methods of calculating the parameters.

13.1 Velocity and Acceleration

Consider the mass m rotating about a fixed point in a circle at the end of a string of length r with a velocity $\vec{V}$ in figure 3.1.

In this system, the radius vector will sweep out an arc $d\theta$ as the mass m moves a distance $ds = rd\theta$ along the circle. The velocity of the mass m will then be[1]

$$V_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \tag{13.1}$$

where $\vec{\omega}$ is defined as the **rotational velocity** with magnitude

$$\omega = \frac{d\theta}{dt}. \tag{13.2}$$

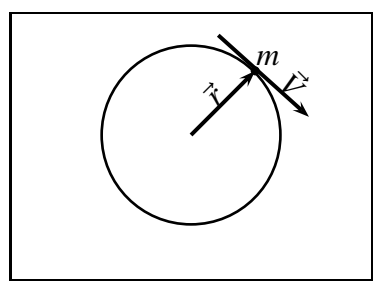


Figure 3.1: Mass rotating in a circle about a point.

[1] In these formulas, angles are measured in radians

The direction of the vector $\vec{\omega}$ is perpendicular to the plane of the drawing and directed into the page in accordance with the right-hand rule.[2]

For the mass to continue moving in a circle, the velocity vector $\vec{V}$ with magnitude defined by equation 13.1 is constrained to be tangential to the circle at the point intersected by the radius vector r. The **rotational acceleration** of the mass about the point is defined as the time rate of change of the rotational velocity.

$$\alpha = \frac{d\omega}{dt} \tag{13.3}$$

A more general and formal explanation of velocity and acceleration in rotational motion is to consider the general case of a mass m moving along a curve as illustrated in figure 3.2.

The equation for the vector defining the position of the mass point can be written in polar coordinates with $\hat{r}$ a unit vector parallel to $\vec{r}$ and $\hat{\theta}$ a unit vector perpendicular to $\vec{r}$ and in the direction of increasing θ.

$$\vec{r} = r\hat{r} \tag{13.4}$$

Upon differentiation, it is seen that the velocity and acceleration vectors have radial and angular components as follows

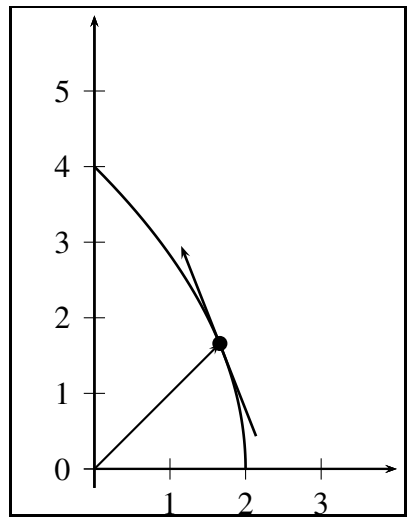


Figure 3.2: Mass point moving along a parabola.

$$\vec{\dot{r}} = \vec{\dot{r}}\hat{r} + r\dot{\theta}\hat{\theta} \tag{13.5}$$

$$\vec{\ddot{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}, \tag{13.6}$$

where the components of velocity and acceleration are

$$V_r = \dot{r} \tag{13.7}$$

$$V_\theta = r\dot{\theta} \tag{13.8}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \tag{13.9}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{13.10}$$

[2]The right hand rule requires that the fingers of the right hand will point in the direction of the tangential velocity when the thumb points in the direction of the vector.

In case the mass m is moving in a circle where the radius vector is constant with $\dot{r} = 0$, these components simplify to

$$V_r = 0 \tag{13.11}$$
$$V_\theta = r\dot{\theta} = r\omega \tag{13.12}$$
$$a_r = r\dot{\theta}^2 = r\omega^2 = \frac{V^2}{r} \tag{13.13}$$
$$a_\theta = r\ddot{\theta} = r\alpha, \tag{13.14}$$

where we have represented the angular velocity and angular acceleration with

$$\omega = \frac{d\theta}{dt} \tag{13.15}$$
$$\alpha = \frac{d\omega}{dt}. \tag{13.16}$$

It should be noted that the velocity vector will always be tangent to the curve. The acceleration vector will be perpendicular to the velocity vector and parallel to the radius vector only when the curve is a circle. The force exerted by the mass m on the string is outward acceleration, referred to as **centrifugal** acceleration, while the force exerted on the mass m by the string is inward acceleration, referred to as **centripetal** acceleration.

As an example of rotational motion, consider a stone weighing 2 ounces embedded in the tread of a automobile tire with the automobile moving at a constant speed of 60 miles/hr or 88 ft/sec. In this case the tangential velocity of the stone must also be 88 ft/sec if there is no slipping. Assume that the tire has an outside diameter of 30 inches or a radius of 1.25 foot. In this case the angular velocity of the radius vector pointing to the stone must be

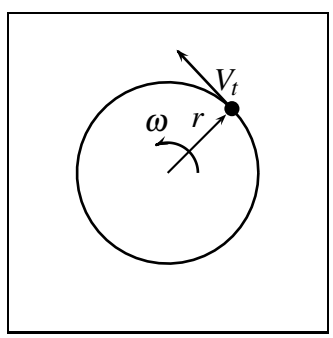


Figure 3.3: Stone embedded in automobile tire.

$$\omega = \frac{V}{r} = \frac{88}{1.25} = 70.4 \quad \text{radians per second} \tag{13.17}$$

The stone will experience a centrifugal acceleration outward of

$$a = r\omega^2 = (1)(70.4)^2 = 5,947 \quad ft/sec^2. \tag{13.18}$$

Since the stone has a weight of 0.125 lb and therefore a mass of 0.0039 $slugs$, the force that must be exerted by the tire on the stone to hold it in place has magnitude

$$F = ma = (0.0039)(5,947) = 23.1 \quad \text{pounds} \tag{13.19}$$

If the stone comes loose from the tire, it will not fly radially outward but follow a path along the direction of the tangent to the tire's surface.

These principles may also be applied to calculate the period of rotation of the moon about the earth. In its orbit about the earth, the moon experiences a gravitational pull toward the earth of $\frac{GMm}{r^2}$ where G is the gravitational constant, M is the mass of the earth, m is the mass of the moon and r is the distance from the center of the earth to the center of the moon. This gravitational pull will be balanced by a centrifugal force of $\frac{mV^2}{r}$, which allows the velocity of the moon in its orbit about the earth to be calculated.

$$\frac{GMm}{r^2} = \frac{mV^2}{r} \quad \text{so that} \tag{13.20}$$

$$V = \sqrt{\frac{GM}{r}} \tag{13.21}$$

Taking $G = 6.67 \times 10^{-11}\ Newton - Meter^2/kilograms^2$, $M = 5.98 \times 10^{24}$ kg and $r = 3.84 \times 10^8$ m the calculated velocity of the moon in orbit is 1.02×10^3 m/sec, which would allow it to traverse its orbit in 27.4 days, comparable to the published period of 27.3 days.

As another example that illustrates the effects of radial acceleration, suppose that an automobile is being driven along a road with a radius of curvature of r feet inclined at an angle of θ degrees and that the coefficient of friction between the automobile tires and the road surface is μ.

The force pushing the car against the roadway is $mr\omega^2$ directed horizontally and the force of gravity pulling the car vertically downward is mg. These two forces provide a combined weight of $mr\omega^2 \sin\theta + mg\cos\theta$ pushing the automobile against the

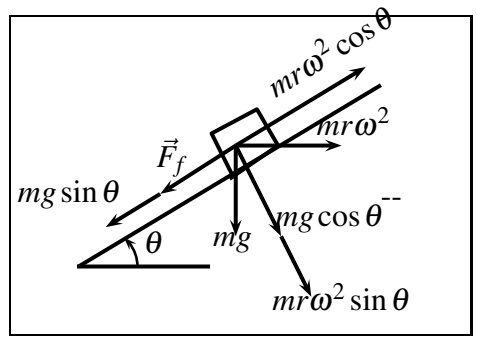


Figure 3.4: Automobile on circular race track.

inclined roadway. As a result, the force that tends to slide the car up the incline is $mr\omega^2\cos\theta$, which must be balanced by the force of gravity $mg\sin\theta$ and the frictional force pulling the automobile down the inclined roadway $\mu\left(mr\omega^2\sin\theta + mg\cos\theta\right)$.

$$mr\omega^2\cos\theta = mg\sin\theta + \mu\left(mr\omega^2\sin\theta + mg\cos\theta\right) \tag{13.22}$$

$$\mu = \frac{mr\omega^2\cos\theta - mg\sin\theta}{mr\omega^2\sin\theta + mg\cos\theta} \tag{13.23}$$

It is apparent that the mass of the automobile cancels out of equation 13.23 so that the force of friction needed to hold the automobile on the road is independent of the mass of the automobile and dependent only on the radius of curvature, the speed of the automobile and the angle of embankment. If the force of friction is not to be counted on to keep the automobile on the roadway, the centrifugal force tending to slide the automobile up the embankment must equal the force of gravity pulling the automobile down the embankment.

$$mr\omega^2\cos\theta = mg\sin\theta \quad \text{therefore} \tag{13.24}$$

$$\tan\theta = \frac{r\omega^2}{g} = \frac{V^2}{gr} \tag{13.25}$$

Thus, the angle of embankment necessary to prevent an automobile traveling 60 miles/hr (88 ft/sec) from sliding up or down the embankment of radius 2400 feet is therefore

$$\theta = \arctan\frac{(88)^2}{(32.17)(2400)} = 5^o 50' \tag{13.26}$$

For the case in which the automobile is a race car, the radius of the curve is 100 feet, the coefficient of friction is 0.60 and the angle of embankment is 30 degrees, the maximum speed at which the car can drive is found from substitution into equation 13.23 to be 185 ft/sec or about 126 mph. Doubling the radius of curvature while keeping the coefficient of friction and angle of embankment the same will multiply the speed at which the car can be driven by $\sqrt{2} = 1.414$.

13.2 Rotational kinematics

By integrating equation 13.15 we can obtain an equation for the angular velocity ω as a function of time very similar to the kinematic equation for linear velocity.

$$\int_{\omega_o}^{\omega} d\omega = \alpha \int_0^t dt \tag{13.27}$$

$$\omega = \omega_o + \alpha t \tag{13.28}$$

Similarly, we can obtain an equation for the angle of rotation θ as a function of time by integrating equation 13.16.

$$\int_{\theta_o}^{\theta} d\theta = \int_0^t \omega dt = \int_0^t (\omega_o + \alpha t)\, dt \tag{13.29}$$

$$\omega = \omega_o + \omega_o t + \frac{1}{2}\alpha t^2 \tag{13.30}$$

By combining these equations together, we can obtain an equation relating the change in angular velocity to the angle through which the system has rotated.

$$\omega^2 = \omega_o^2 + 2\alpha\,(\theta - \theta_o) \tag{13.31}$$

13.3 Moment of Inertia

The **center of mass** of a collection of mass points can be defined as the average of their mass weighted positions as illustrated in figure 3.5. For practical purposes the center of mass is the same as the **center of gravity** except in problems of planetary motion in which the acceleration of gravity may vary.

By its definition then, the center of mass for a collection of mass points m_i is located with the vector $\vec{r}$

$$\mathbf{r} = \frac{\sum m_i r_i}{\sum m_i} \tag{13.32}$$

Using the same approach, we can define the center of mass of a continuous distribution with mass density $\rho(r)$ and total mass M distributed over a volume V as follows

$$\mathbf{R} = \frac{1}{M}\int \rho(r) r dV \tag{13.33}$$

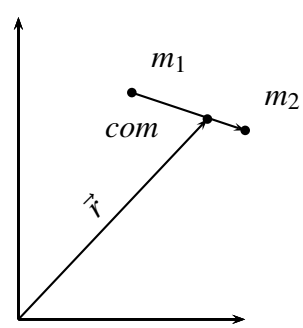


Figure 3.5: Center of Mass

Consider next a collection of N mass points m_i each rotating with the same angular velocity ω around an axis so that each mass point has a tangential velocity $v_i = r_i\omega_i$. The total kinetic energy of this collection of mass points will be

$$K = \sum_1^N \frac{1}{2} m_i V_i^2 = \frac{1}{2}\omega^2 \sum_1^N m_i r_i^2 = \frac{1}{2} I \omega^2 \quad \text{where} \tag{13.34}$$

$$I = \sum_1^N m_i r_i^2 \tag{13.35}$$

The quantity I, known as the **moment of inertia**, depends only upon the geometric distribution of the mass density and may be generalized to a solid body by integrating over the volume.

$$I = \int_V \rho(r) r^2 dV \tag{13.36}$$

The definition of moment of inertia is referenced to a specific axis of rotation. If the axis passes through the center of mass, the moment of inertia may be easily calculated by integration for regular bodies such as spheres and ellipsoids. The moments of inertia for several regular bodies are listed in table 3.6

Object	Moment of Inertia
Solid cylinder of radius r about longitudinal axis	$I = \frac{1}{2}mr^2$
Hollow cylindrical shell of radius r about longitudinal axis	$I = mr^2$
Sphere of radius r about axis through its center	$I = \frac{2}{5}mr^2$
Spherical shell of radius r about axis through its center	$I = \frac{2}{3}mr^2$
Slender rod of length L about axis through its center	$I = \frac{1}{12}mL^2$
Elliptic cylinder of semiaxes a and b about longitudinal axis	$I = \frac{1}{4}m\left(a^2 + b^2\right)$
Right circular cone of base radius r about axis of revolution	$I = \frac{3}{10}mr^2$

Figure 3.6: Moments of Inertia for regular bodies

As an example of finding moments of inertia by integration over a volume, consider the right circular cylinder of mass m illustrated in figure 3.7. Using equation 13.36 and taking the differential element of volume to be $2\pi t\ell dt$ and the mass of the cylinder to be $M = \rho\pi r^2\ell$.

.

$$I = \int_0^r \rho(t^2)(2\pi\ell t dt) = \frac{1}{2}\pi\ell\rho r^4 = \frac{1}{2}Mr^2 \tag{13.37}$$

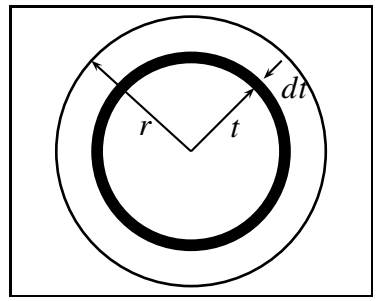


Figure 3.7: Moment of inertia for right circular cylinder.

If the moment of inertia I is known for a certain axis, the **parallel axis theorem** in mathematics provides a simple formula for the moment of inertia I_d about another parallel axis located a distance d away.

$$I = I + md^2 \tag{13.38}$$

This formula only works in the case of parallel axes. For non-parallel axes the orientation of each axis must be taken into account.

Since the moment of inertia about an axis through a body can be written as the sum of mass elements times the square of the distance of those mass elements from the axis of rotation, the moment of inertia may be expressed as the product of the total mass and the square of a distance k which is known as the **radius of gyration**.

$$I = \sum_i m_i r_i^2 = Mk^2 \quad \text{where} \tag{13.39}$$

$$M = \sum_i m_i \tag{13.40}$$

Physically, the radius of gyration may be considered the distance from the axis of rotation to the point at which all the mass may be considered to be concentrated to get the same moment of inertia. The radius of gyration is useful in calculating the moment of inertia of irregular shaped bodies. A transfer formula similar to equation 13.38 also exists for the radius of gyration.

$$k = k + d^2 \tag{13.41}$$

13.4 Kinetic Energy

The mass point moving along the curve will have kinetic energy and momentum in the same way as for non-rotational motion, which may be defined as follows

for a mass point moving in a circle.

$$K = \frac{1}{2}mr^2\omega^2 \tag{13.42}$$
$$p = mr\omega \tag{13.43}$$

For solid bodies having a moment of inertia I, the rotational kinetic energy may be written

$$K = \frac{1}{2}I\omega^2. \tag{13.44}$$

13.5 Angular Momentum

The **angular momentum** of a mass point constrained to rotate about a fixed axes in a circle of radius r is defined by

$$\vec{L} = \vec{r} \times \vec{p} \tag{13.45}$$

where $\vec{p}$ is the linear momentum and has magnitude $mVr\sin\theta$. The vector $\vec{L}$ is perpendicular to the plane of motion defined by $\vec{r}$ and $\vec{p}$ with direction defined by the right-hand rule. In the case of motion in a circle where the velocity vector is tangent to the circle, the magnitude of the angular momentum becomes

$$L = mr^2\omega \tag{13.46}$$

Bodies of solid mass and finite dimensions also have angular momentum which may also be defined in terms of its moment of inertia.

$$L = I\omega \tag{13.47}$$

13.6 Torque

In case of circular motion, a force F acting on a mass point constrained to move along a circle at a point P defined by the vector r will produce a **torque** τ equal to the vector cross-product of r and F perpendicular to the plane of r and F and having direction in accordance with the right hand rule.

$$\vec{\tau} = \vec{r} \times \vec{F} \quad with\, magnitude \tag{13.48}$$
$$|\vec{\tau}| = rF\sin\theta \tag{13.49}$$

Using Newton's law for the magnitude of the force allows the torque to be written in terms of the tangential angular acceleration.

$$\tau = mra = mr^2\alpha \tag{13.50}$$

In the case of a solid mass of finite dimensions, the torque can be written in terms of the moment of inertia.

$$\tau = \int mr^2\alpha dV = I\alpha \tag{13.51}$$

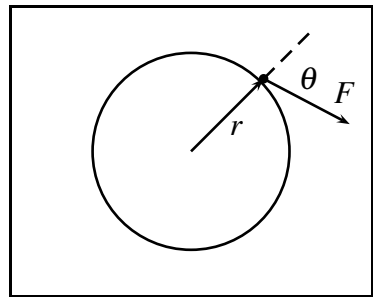


Figure 3.8: Torque due to force F acting on a rotating mass point.

It is important to note that the torque is the time rate of change of angular momentum.

$$\tau = \frac{dL}{dt} \tag{13.52}$$

Therefore, If the net torque acting on the body is zero, the angular momentum will be **conserved**. This important conservation law is the rotational parallel to the linear conservation law which states that if the linear force acting on a particle is zero, the linear momentum is conserved.

Finally, the work done by a torque acting through an angle $\Delta\theta$ is $W = \tau\Delta\theta$ so that the power delivered by a torque becomes

$$P = \tau\omega \tag{13.53}$$

13.7 Comparison of Rotational and Linear Formulas

A side-by-side comparison of rotational formulas to non-rotational formulas proves helpful in understanding the parallel nature of these two disciplines as illustrated in table 3.1 below.

Problems

40. In 1934 J.W. Beams and co-workers at the University of Virginia showed that it was feasible to separate isotopes of Uranium in a gas centrifuge. If a centrifuge is rotating at 20,000 rps and has a radius of 0.500 m. What

Linear kinematics	Rotational kinematics
$V = \frac{ds}{dt}$ $a = \frac{dV}{dt}$	$\omega = \frac{d\theta}{dt}$ $\alpha = \frac{d\omega}{dt}$
$V = V_o + at$ $s = s_o + V_{os}t + \frac{1}{2}at^2$ $V^2 = V_o^2 + 2a(s - s_o)$	$\omega = \omega_o + \alpha t$ $\omega = \omega_o + \omega_o t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$
$K = \frac{1}{2}mV^2$ $p = mV$	$K = \frac{1}{2}I\omega^2$ $L = I\omega$
$F = ma$	$\tau = I\alpha$
$F = \frac{dp}{dt}$	$\tau = \frac{dL}{dt}$
$P = FV$	$P = \tau\omega$

Table 3.1: Comparison of Linear and Rotational Kinematic formulas.

is the angular velocity, the centrifugal acceleration and centrifugal force on a molecule of Uranium hexafluoride, UF_6 located near the outer perimeter of the centrifuge . Take the molecular weight of UF_6 to be 5.845×10^{-22} grams. ans. 1.257×10^5 rad/sec, 7.896×10^9 m/sec^2 and 4.615×10^{-15} N

41. An airplane pilot can withstand an acceleration of about 9g without loosing consciousness. If his airplane is moving at 180 m/sec, what is the radius of curvature of the tightest turn he can make? ans. 367 meters.

42. A pendulum bob has mass of 0.100 kg and the pendulum arm has length of 0.300 meters. If the pendulum is pulled back to an angle of 37.0 degrees with the vertical, what will be the velocity of the pendulum bob as it passes through its lowest point, the angular acceleration, centrifugal acceleration and the centrifugal force it exerts on the pendulum arm? ans. 1.089 m/sec, 3.63 rad/sec, 3.95 m/sec^2, 0.395 N.

43. What would be the radial acceleration of a race car rounding a curve in a race track of 10.0 meter radius and moving at 30.0 m/sec? ans. 90.0 m/sec^2

44. Consider two balls each of mass 0.100 kg fixed to each end of a rigid rod 1.00 meter long rotating about a perpendicular axis through its center at an angular velocity of 135.6 radians/sec. Calculate the tangential velocity and radial acceleration of the balls, the moment of inertia, angular momentum

and kinetic energy of the system. ans. 6.80 m/sec, 92.5 m/sec^2, 0.500 $kg-m^2$, 0.680 $N-m$, 92.5 Joules

45. A 0.500 kg mass is attached to a string which is wound around a pulley or radius 10.0 cm having a moment of inertia of 2.00 $kg-m^2$ and turned loose. What is the angular acceleration of the pulley, the torque applied to the pulley, and the linear acceleration of the system? After the 0.500 kg mass falls for 2.00 seconds, what is the angular velocity of the pulley, the linear velocity of the 50.0 gm mass, the angular momentum of the pulley, the kinetic energy of the pulley, the kinetic energy of the mass and the kinetic energy of the system. ans. 0.2395 rad/sec^2, 0.479 N-m, 0.02395 m/sec^2; 0.479 rad/sec, 0.0479 m/sec, 0.958 $N-m$, 0.2294 Joules, 0.000574 Joules, 0.2299 Joules

46. Compute the moment of inertia of a uniformly dense sphere of total mass M and radius a about a diameter lying along the Y-axis. ans. $I = \frac{2}{5}Ma^2$

47. Determine the moment of inertia and radius of gyration with respect to its major axis for an ellipsoid having equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with $a=b$. ans. $I_z = \frac{2}{5}Mb^2$, $k_z = \sqrt{\frac{2b^2}{5}}$.

14 Application to physical systems

Formulas presented in this chapter can be used to analyze and design numerous mechanical systems of use in physics and engineering. Several of these applications are presented in the following examples.

14.1 Engine Governor

Rotation of the system in figure 3.9(A) is driven by an engine crankshaft. For a fixed length of the arms, the angle at which the two arms make with the verticle drive increases as the angular velocity ω of the drive shaft increases. By allowing the governor to restrict or increases the opening through which the fuel mixture flows, the rpm of the engine can be controlled.

$$mg\sin\theta = mr\omega^2\cos\theta \quad \text{so that} \tag{14.1}$$

$$\tan\theta = \frac{r\omega^2}{g} \tag{14.2}$$

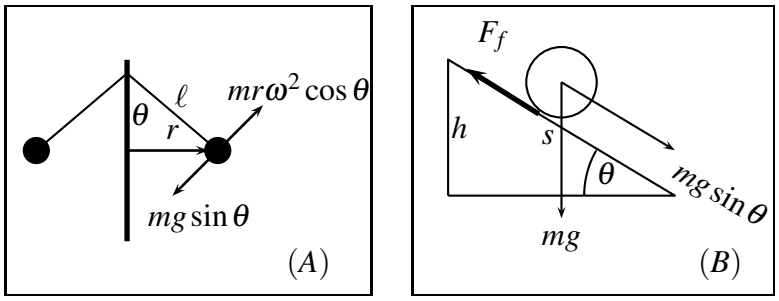


Figure 3.9: Governor controlling rpm of engine (A) and cylinder rolling down an incline (B).

14.2 Cylinder Rolling down Incline

The effect of the moment of inertia becomes more evident as we calculate the angular acceleration and velocity of objects rolling down an incline. The two equations that describe the rolling of a circular object may be written as follows where the first equates the unbalanced force acting down the incline to the product of mass and acceleration and the second represents the torque about the rolling axis caused by the frictional force.

$$mg\sin\theta - f = ma = mr\alpha \tag{14.3}$$

$$fr = I_o\alpha \tag{14.4}$$

These equations may be combined to obtain an expression for the angular acceleration of the object down the incline.

$$\alpha = \frac{mgr\sin\theta}{I_o + mr^2} \tag{14.5}$$

It is apparent that the acceleration of the object depends on the moment of inertia about the rolling axis. Results are tabulated for a solid cylinder, cylindrical shell and solid spherical ball in table 3.2

Object	Moment of Inertia	Acceleration
Solid right circular cylinder	$I = \frac{1}{2}mr^2$	$a = \frac{2}{3}g\sin\theta$
Hollow cylindrical shell	$I = mr^2$	$a = \frac{1}{2}g\sin\theta$
Solid Sphere	$I = \frac{2}{5}mr^2$	$a = \frac{5}{7}g\sin\theta$

Table 3.2: Moments of Inertia for regular bodies

This example explains why objects of the same radius but different composition roll down the inclined plane with different accelerations.

The same results could have been obtained using conservation of energy. At the top of the incline, the potential energy of the object is mgh, where h is the height of the incline, and the kinetic energy is zero giving a total energy equal to the potential energy. After the object has rolled a distance $s = h/\sin\theta$ down the incline, the potential energy will be zero while the kinetic and total energy is $\frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$. Equating the energy at the top and bottom results in an equation for the angular velocity of the object at the bottom of the incline.

$$mgh = \tfrac{1}{2}mV^2 + \tfrac{1}{2}I\omega^2 \quad \text{so that} \tag{14.6}$$

$$\omega^2 = \frac{2mgh}{I_o + mr^2} \tag{14.7}$$

Using equation 13.31 and equating $\theta - \theta_o$ to $s/2\pi r = 2\pi h/r\sin\theta$ gives for the angular acceleration

$$\alpha = \frac{mgr\sin\theta}{I_o + mr^2} \tag{14.8}$$

and the linear acceleration

$$a = r\alpha = \frac{mgr^2\sin\theta}{I_o + mr^2} = \frac{5}{7}g\sin\theta \tag{14.9}$$

From this, it is easy to see that when a solid sphere, a solid cylinder and a thin hollow cylinder roll down an incline together that the solid sphere rolls faster than either of the other objects and that the solid cylinder rolls faster than the hollow cylinder.

14.3 Falling and Rotating Objects

A cylinder rolling down a line under the influence of gravity will accelerate at a rate dependent upon its moment of inertia. This dependence can be illustrated by calculating the angular acceleration of the cylinder in figure **??**

The following two equations describe the falling and rotating action of the cylinder.

$$mg - T = ma = mr\alpha \tag{14.10}$$

$$Tr = I_o\alpha \tag{14.11}$$

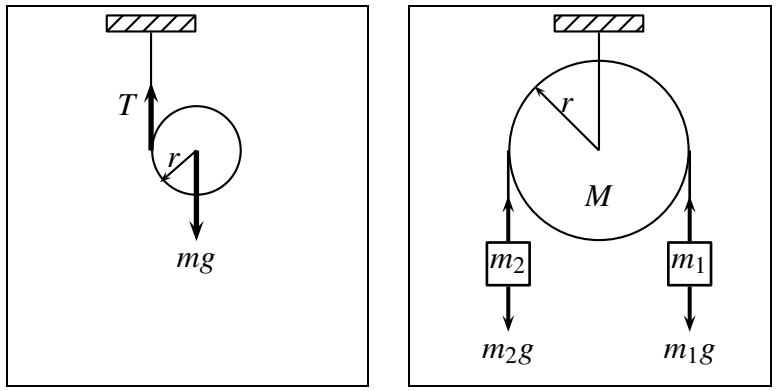


Figure 3.10: Cylinder rolling down incline (A) and Atwood machine (B).

Combining these equations and solving for the angular acceleration gives us

$$\alpha = \frac{mr}{I_o + mr^2} \tag{14.12}$$

which shows that objects of different size, composition and moment of inertia will fall at different rates.

14.4 Atwood Machine

The Rev. George Atwood invented the Atwood machine in 1784 as a laboratory experiment to verify the mechanical laws of motion with constant acceleration. It has become a traditional fixture in almost all physics laboratories and is a simple system consisting of a string attached to two masses m_1 and m_2 and passing over a pulley with moment of inertia I.

In figure 3.10(A) two masses m_1 and m_2 are connected by a string that is wound around a pulley of radius r and moment of inertia I_o. Three equations are needed to determine the direction of rotation of the pulley and the acceleration of the masses. We will assume m_1 is greater than m_2 so that motion is clockwise.

$$m_1g - T_1 = m_1a = m_1R\alpha \tag{14.13}$$

$$T_2 - m_2g = m_2a = m_2R\alpha \tag{14.14}$$

$$RT_1 - RT_2 = I_o\alpha \tag{14.15}$$

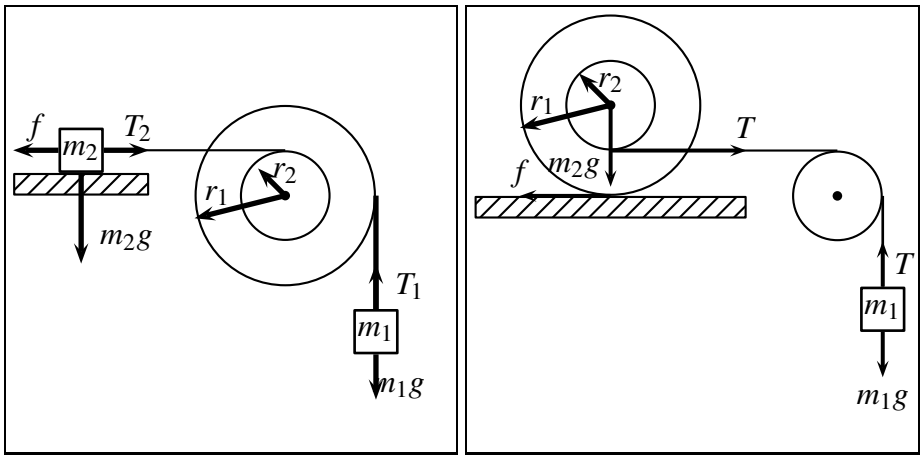


Figure 3.11: Pulling a block across the table (A) and pulling a compound cylinder across the table.

Combining and solving for α gives

$$\alpha = \frac{(m_1 - m_2)gR}{I_o + m_1R^2 + m_2R^2} \tag{14.16}$$

14.5 Pulling a Block Across Table

When a block of mass m_2 is placed on a table and attached to a pulley being rotated by the weight of another block of mass m_1 the acceleration of the pulley is retarded by the force of kinetic friction.

The equations needed to relate all the variables in this example are

$$m_1g - T_1 = m_1a_1 = m_1r_1\alpha \tag{14.17}$$

$$T_2 - \mu m_2g = m_2a_2 = m_2r_2\alpha \tag{14.18}$$

$$r_1T_1 - r_2T_2 = I_o\alpha \tag{14.19}$$

The frictional force f has been set equal μm_2g in these equations. Combining and solving for α gives

$$\alpha = \frac{(m_1r_1 - \mu m_2r_2)g}{I_o + m_1r_1^2 + m_2r_2^2} \tag{14.20}$$

The coefficient of static friction μ_s required to prevent motion from starting can be obtained by requiring the numerator in equation 14.20 to be zero.

$$\mu = \frac{m_1 r_1}{m_2 r_2} \tag{14.21}$$

14.6 Pulling a Compound Cylinder across a Table

For this problem, illustrated in figure 3.11(B), we will take the mass and moment of inertia of the pulley over which the mass m_1 hangs to be zero. The tension in the string will cause the cylinder to roll clockwise but this rotation will roll up the string working against the force of gravity tending to pull the mass m_1 downward. We also require that the coefficient of static friction is sufficient to prevent the cylinder from slipping. These forces are described by the equations

$$m_1 g - T = m_1 a_1 \tag{14.22}$$

$$T - f_s = m_2 a_2 \tag{14.23}$$

$$f_s r_1 - T r_2 = I_o \alpha \tag{14.24}$$

Noting that $a_2 = r_1 \alpha$ and combining the second two equations gives the angular acceleration of the cylinder,

$$\alpha = \frac{r_1 - r_2}{I_o + m_2 r_1^2} T. \tag{14.25}$$

Taking T from the first equation and noting that $a_1 = (r_1 - r_2)\alpha$ provides a formula for the angular acceleration without the tension in the string.

$$\alpha = \frac{m_1 (r_1 - r_2) g}{I_o + m_2 r_1^2 + m_1 (r_1 - r_2)^2} \tag{14.26}$$

From this equation it is seen that when $r_2 = 0$, $a_1 = a_2$ and when $r_1 = r_2$ the cylinder will not move unless slipping occurs.

14.7 The Yo-Yo Problem

The "Yo-Yo" was known to be in existence in 449 BC and has been used throughout the millenia.[3] You have no doubt often wondered how the simple Yo-Yo

[3] James L. Haven and Charles Hettrick of Cincinnati, Ohio, USA, received the first US patent on the Yo-Yo in 1866.

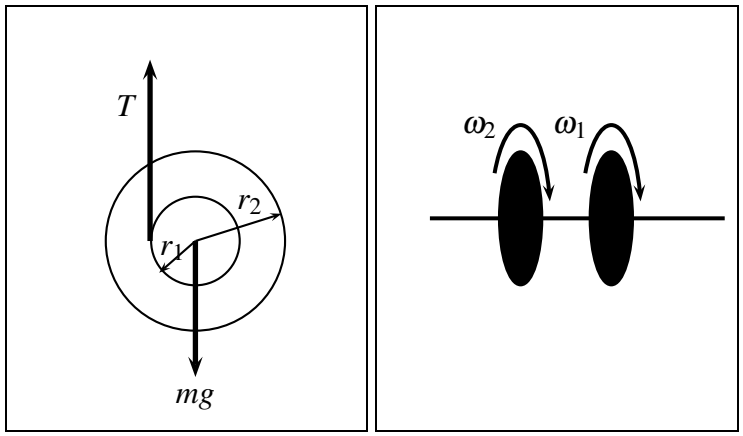


Figure 3.12: The Yo-Yo problem (A) and mechanical clutch (B).

worked. Now, we are in a position to explain why the Yo-Yo climbs back up the string as a simple consequence of energy conservation. Consider the diagram in figure 3.12(A). The force of gravity pulls the yo-yo downward while the tension in the string pulls it upward. The unbalanced torque causes it to rotate. These statements are reflected in the equations below.

$$mg - T = ma = mr_1\alpha \tag{14.27}$$

$$Tr_1 = I_o\alpha \tag{14.28}$$

Noting that $a = r_1\alpha$, combining these two equations and solving for α we find

$$\alpha = \frac{mr_1}{I_o + mr_1^2}g \quad \text{and} \tag{14.29}$$

$$a = \frac{1}{1 + \frac{I_o}{mr_1 2}}g. \tag{14.30}$$

When the yo-yo is a the top of its trajectory, its kinetic energy is zero and the potential energy mgh equals the total energy. As the yo-yo falls downward, it looses potential energy and gains kinetic energy, which is shared between rotational and translation motion. At the bottom of its trajectory, its potential energy is zero and the kinetic energy is rotational since the translational motion downward has stopped. The total energy is still constant at the value mgh and the yo-yo begins to roll up the back side of the string this time converting kinetic energy to

potential energy until it has reached the point where it started. Numerous tricks are possible with the yo-yo. These can all be explained with mechanical equations.

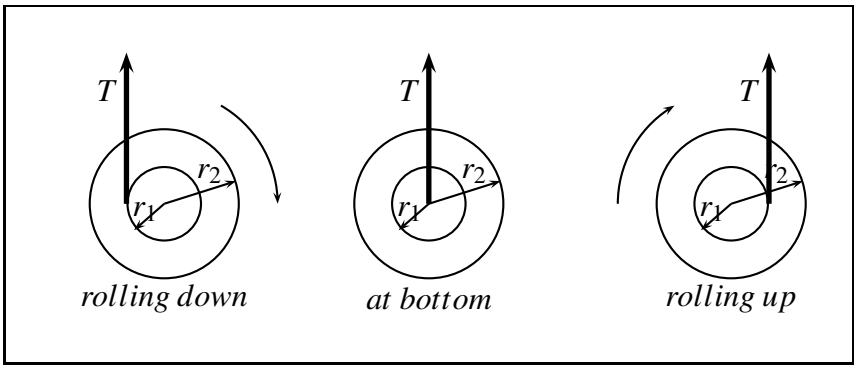


Figure 3.13: Rotation of Yo-Yo rolling down string and climbing up string.

14.8 Mechanical Clutch

As an example of conservation of angular momentum, consider the clutch assembly of figure 3.12(B) in which two disks are mounted on a shafts that can be connected by a clutch. Disk 1 has a moment of inertia I_1 and rotates with an angular velocity ω_1 while disk 2 has a moment of inertia I_2 and rotates with an angular velocity ω_2. If there are no external torques acting on the system, the angular velocity after the clutch is engaged is

$$I_1\omega_1 \pm I_2\omega_2 = (I_1 + I_2)\,\omega \tag{14.31}$$

$$\omega = \frac{I_1\omega_1 \pm I_2\omega_2}{(I_1 + I_2)} \tag{14.32}$$

The sign is positive if the pulleys are initially rotating in the same direction and negative if otherwise.

14.9 Drive Pulley

A drive belt from an engine shaft A is wrapped around a drive pulley B to produce a torque in the pulley shaft as illustrated in figure 3.14. Slippage of the belt on the drive pulley is prevented by a coefficient of friction μ. As a result, a tension T_1 develops on the tight side of the belt and T_2 on the slack side. Recalling that the

tension vector in the belt will be tangent to the circumference of the drive pulley and taking a radius vector centered in an arc of width $d\theta$, it is seen that the angle between two tangent lines, one, T, at the intersection of the radius vector and the other, $T+dT$, at the intersection of a radius vector at $\theta+\frac{1}{2}d\theta$, will be $\frac{1}{2}d\theta$. Then, resolving the components of the tensions T and $T+dT$ into components normal to the surface it seen that the normal force forcing the belt against the drive pulley is

$$N=(T+dT)\sin\frac{1}{2}d\theta+(T)\sin\frac{1}{2}d\theta\simeq Td\theta$$

where we have taken $2T+dT\simeq 2T$ and $\sin\frac{\theta}{2}\simeq\frac{\theta}{2}$, so that the force of friction is

$$f=\mu Td\theta. \tag{14.33}$$

Since the incremental difference between the torques produced by the tension in the belt is counterbalanced by the torque due to the force of friction, we can write

$$(T+dT)r-Tr=\mu Trd\theta, \tag{14.34}$$

from which we obtain

$$rdT=\mu Trd\theta. \tag{14.35}$$

Rewriting this equation gives

$$\frac{dT}{T}=\mu d\theta, \tag{14.36}$$

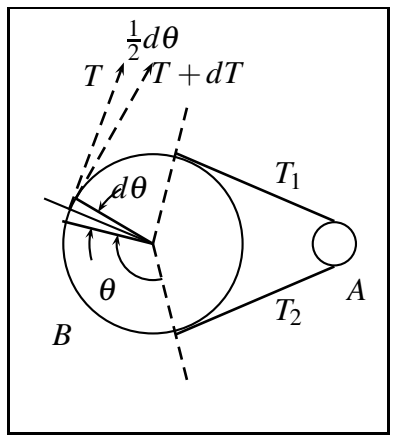


Figure 3.14: Drive pulley.

which can be integrated from T_2 to T_1 and from $\theta=0$ to $\theta=\beta$

$$\int_{T_2}^{T_1}\frac{dT}{T}=\int_0^{\beta}\mu d\theta \tag{14.37}$$

to obtain the relation between T_1 and T_2

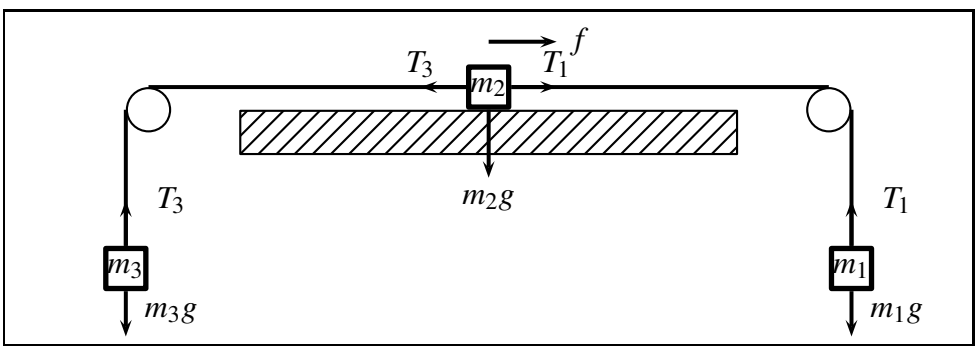
$$T_1=T_2e^{\mu\beta}. \tag{14.38}$$

The torque developed in the drive pulley is then

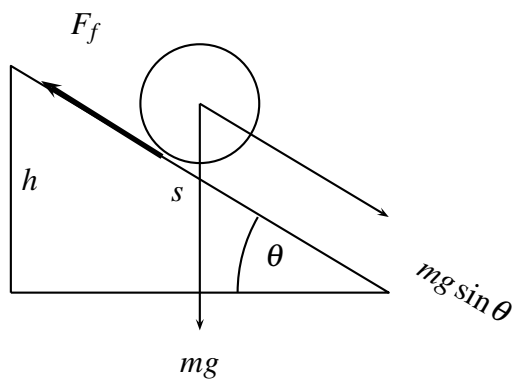
$$\tau=(T_1-T_2)\,r=\left(e^{-\mu\beta}-1\right)T_1r \tag{14.39}$$

Problems

48. A block m_2 weighing 5.00 kg is placed on a table where the coefficient is 0.500 for static friction and 0.250 for kinetic friction. A line from one side passes over a massless, frictionless pulley to a block m_1 weighing 2.00 kg while a line from the other side passes over a massless, frictionless pulley to a block m_3 weighing 8.00 kg. What is the force of kinetic friction and what is the maximum force that static friction can develop? Will the system move without a starting impulse? After starting which way will the system move, what is the acceleration of the blocks and what is the tension in the two lines? ans. $F_k = 12.26$ N, $F_s = 24.53$ N, Yes, Left, $a = 3.11\ m/sec^2$, $T_8 = 53.6$ N, $T_2 = 13.4$ N.



49. A solid steel ball of mass $M = 263$ g and radius $a = 0.020$ m is released at the top of an incline of height $h = 1.00$ m to roll without slipping down a board inclined at an angle of 54.0-degrees with respect to the horizontal. Compute the linear and angular accelerations as the ball rolls down the incline, the linear and angular velocities at the bottom of the incline, the potential and total energies at the top of the incline and the linear and rotational kinetic energies and the angular momentum of the ball when it reaches the bottom of the incline. ans. $a = 5.67\ m/sec^2$, $\alpha = 283$ radians/sec, $V = 3.74$ m/sec, $\omega = 187$ rad/sec, $U = E = 2.58$ J, $K_{linear} = 1.84$ J, $K_{rot} = 0.737$ J, $L = 0.00788\ kg - m^2/sec$.

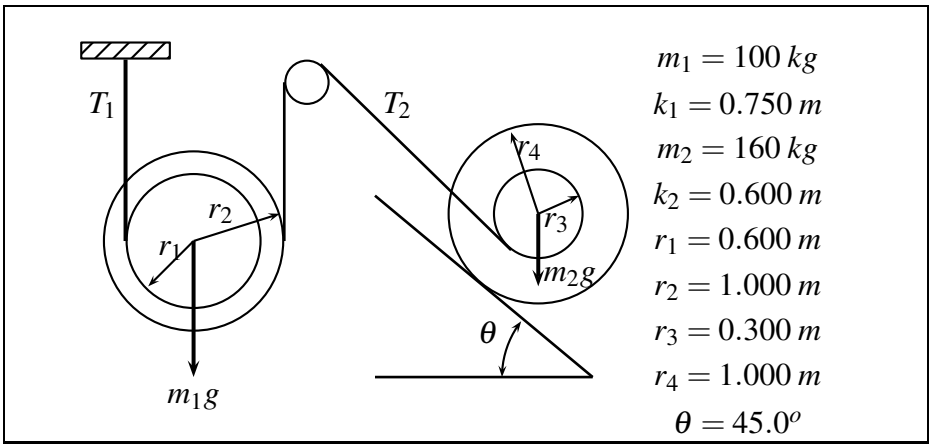


(B)

50. In the figure below, pulley 1 is supported by a line wound around the inside drum. A second line wound around the outside drum passes over a massless, frictionless pulley and connects to the inner drum of pulley 2. Compute the moment of inertia of pulley 1 about the point at which the support line is wound around the inside drum and the moment of inertia of pulley 2 about the point where the outside drum touches the inclined plane. ans. $I_1 = 92.25$ $N-m/sec^2$, $I_2 = 217.6$ $N-m/sec^2$ Then compute the kinematic relation between the angular acceleration of pulley 1 and that of pulley 2 by assuming that the two pulleys are rotating about instant centers at their points of contact with the support line and inclined plane. ans. $\alpha_2 = \frac{1.6}{0.7}\alpha_1$ Next, compute the angular accelerations α_1 and α_2 in each pulley and the tension in the supporting line T_1 and the line attached to pulley T_2 by writing equations for the moments about each point of contact to eliminate the unknown frictional force and one of the tensions from the equations. ans. $\alpha_1 = 1.585$ rad/sec^2, $\alpha_2 = 3.623$ rad/sec^2, $T_1 = 616.84$ N, and $T_2 = 459.27$ N.

51. A freight train weighing 1600 tons has rolling resistance of 12 pounds per ton. A 6,000 hp locomotive pulls the train up a 2% grade. What is the maximum speed that the locomotive can develop pulling the train up the grade? ans 27 mph

52. A boat's hawser is wrapped about a capstan on the dock. The coefficient of friction is 0.30. If the boat exerts a pull of 4000 lb on the hawser, how many turns of the hawser will have to be made around the capstan so that the force needed to hold the hawser on the other end does not exceed 50 lb? ans. 2.33 times



15 Pendulums

The pendulum plays such an important part in every aspect of daily life that a more extensive treatment is necessary. Several types, their properties and uses will be considered in this section.

15.1 Clock Pendulum

The Dutch scientist Christian Huygens, inspired by an unfinished design by Galileo Galilei, is credited with invention of the pendulum clock in 1656. His invention improved the accuracy of clocks form 15 minutes per day to around 15 seconds per day. Referring to figure 3.15 we see that a force of gravity mg pulls the pendulum bob downward creating a force $mg\sin\theta$ tangent to the arc of the pendulum bob. This force results in a torque that may be written as follows.

$$\ell\ddot{\theta} = -g\sin\theta \quad \text{or} \tag{15.1}$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \tag{15.2}$$

The function $\sin\theta$ can be expanded in the series

$$sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \cdots \tag{15.3}$$

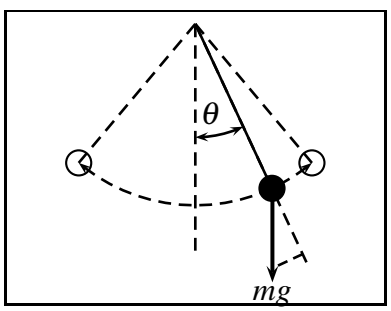


Figure 3.15: Simple Pendulum Clock.

For small angles, and when the angle is measured in radians, the $\sin\theta$ can be approximated by the angle itself in which case the differential equation of motion becomes.[4]

$$\ddot{\theta} + \omega^2\theta = 0 \quad \text{where} \tag{15.4}$$

$$\omega = \sqrt{\frac{g}{\ell}} \tag{15.5}$$

with solution

$$\theta = \theta_o \sin\omega t \tag{15.6}$$

This is the equation of a simple oscillator of amplitude θ_o and period

$$T = 2\pi\sqrt{\frac{\ell}{g}} \tag{15.7}$$

From this equation it is clear that the period of the simple clock pendulum is not dependent on the mass of the pendulum bob but sensitive to the acceleration of gravity and length of the pendulum arm. Therefore, the length of the pendulum arm could be adjusted to control the period. It is also clear that the pendulum can be used to measure the acceleration of gravity. Refinements between the years 1673 and 1821 reduced the error of the pendulum clock to a few seconds per week and made possible studies of the shape of the earth and more accurate measurements of longitude.

15.2 Conical Pendulum

The conical pendulum, first studied by the English scientist Robert Hooke in 1660 and by the Dutch scientist Christiaan Huygens in 1673, consists of a pendulum bob at the end of a string set in rotation about the vertical axis. In its stable orbit, the downward force of gravity is balanced by the vertical component of the tension in the string while the centrifugal force pulling the pendulum outward is balanced by the horizontal component of the tension in the string.

$$T\sin\theta = mg \tag{15.8}$$

$$T\cos\theta = mr\omega^2 \tag{15.9}$$

[4]This approximation has a 2% error at 30-degrees.

Eliminating the tension and mass from these equations gives an expression for the angular velocity.

$$\omega = \sqrt{\frac{g}{r\tan\theta}}, \tag{15.10}$$

This expression can be used to obtain the period of the pendulum after substituting $L\cos\theta$ for r.

$$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}. \tag{15.11}$$

It may be noted that for small angles $\cos\theta \simeq 1$, and the period for the conical pendulum becomes the same as that for the simple pendulum.

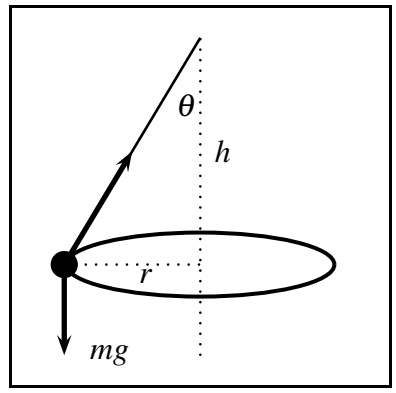


Figure 3.16: Conical Pendulum.

15.3 Torsion Pendulum

The torsion pendulum consists of an object, usually a disk or thin rod, suspended by a thin wire from a rigid support. The object or disk is rotated so as to twist the wire and released. When released, the disk begins to oscillate though an angle θ. The torque exerted on the disk by the wire is proportional to the angle of rotation,

$$\tau = -\kappa\theta. \tag{15.12}$$

The torque may also be expressed in terms of the angular acceleration of the disk.

$$\tau = I\alpha = -I\ddot{\theta}. \tag{15.13}$$

Equating the right hand side of each equation gives the equation of motion for the disk,

$$\ddot{\theta} + \omega^2\theta = 0 \quad where \tag{15.14}$$

$$\omega = \sqrt{\frac{\kappa}{I}} \tag{15.15}$$

It is immediately obvious that the solution to this equation is

$$\theta = \theta_o \sin\omega t, \tag{15.16}$$

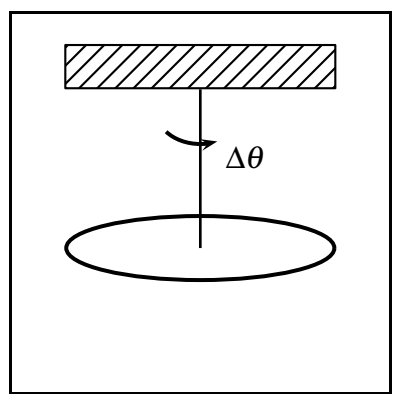


Figure 3.17: Torsion Pendulum.

and that the period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{15.17}$$

Torsion bars are found in many instruments. For example, the balance wheel in a mechanical wristwatch is a torsion pendulum in which the restoring torque is provided by a coiled spring. Two of its most well-known uses were by Charles-Augustin de Coulomb in 1777 to measure the electrostatic force between charges to establish Coulomb's Law, and by Henry Cavendish in 1798 to measure the gravitational force between two masses which led to a value for the gravitational constant. There are at least three types of torsion bars, one is a straight bar of metal or rubber that is subjected to twisting (shear stress) about its axis by torque applied at its ends; a second and more delicate form used in sensitive instruments is a torsion fiber made from silk, glass, or quartz; and a third less delicate form is a helical torsion spring, often used in mouse traps. Credit for its invention is usually credited to Charles-Augustin de Coulomb (1777); although independent claims were recorded by Robert Leslie (1793) and Aaron Crane (1841), who made clocks that would run up to one year on one winding, and the German Anton Harder (1879).

15.4 Compound pendulum

In figure 3.18 a rigid body is pivoted about a point at P a distance b from its center of gravity C and is free to rotate about an axis through P. The body is displaced by an angle θ from the vertical position and allowed to oscillate. The equation of motion for the body is

$$I\ddot{\theta} = -mgb\sin\theta, \tag{15.18}$$

where I is the moment of inertia about an axis through the point P. This equation is solved by

$$\theta = A\sin\omega t, \tag{15.19}$$

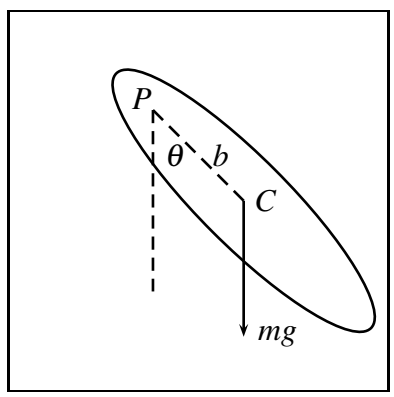


Figure 3.18: Compound Pendulum.

where the angular frequency and period are defined by

$$\omega = \sqrt{\frac{mgb}{I}} \tag{15.20}$$

$$T = 2\pi\sqrt{\frac{I}{mgb}}. \tag{15.21}$$

The rigid body rotating about a fixed axis is thus equivalent to a simple pendulum of length $L = I/mb$.

15.5 Damped Pendulum Oscillations

If the pendulum oscillates in a resistive medium such as water, the motion will be damped and the oscillation amplitude will decrease toward zero. To understand damping of oscillations, suppose that the resistive force is directly proportional to the velocity of the pendulum bob so that the equation of motion becomes

$$\ddot{\theta} + k\dot{\theta} + \omega_o^2\theta = 0 \tag{15.22}$$

where $\omega_o^2 = \frac{g}{\ell}$. This equation has the general solution

$$\theta = e^{-\left(\frac{k}{2}\right)t}\left[Ae^{+\left(\left(\frac{k}{2}\right)^2-\omega_o^2\right)^{1/2}} + BAe^{-\left(\left(\frac{k}{2}\right)^2-\omega_o^2\right)^{1/2}}\right] \tag{15.23}$$

If the constant $\omega_o^2 > \left(\frac{k}{2}\right)^2$, the oscillation will be **underdamped** and the amplitude of the oscillations will decrease toward zero. In this case the oscillations can be described by

$$\theta = Ce^{-\left(\frac{k}{2}\right)t}\cos\omega_1 t \quad \text{where} \tag{15.24}$$

$$\omega_1^2 = \omega_o^2 - \left(\frac{k}{2}\right)^2 \tag{15.25}$$

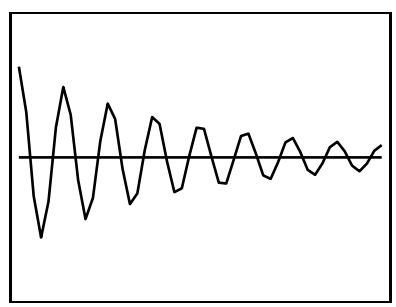


Figure 3.19: Underdamped oscillations.

If the constant $\omega_o^2 = \left(\frac{k}{2}\right)^2$, the oscillation will be **critically damped**, and if the constant $\omega_o^2 > \left(\frac{k}{2}\right)^2$, the oscillation will be **overdamped**. In both cases the amplitude of the pendulum bob swing will decrease toward zero exponentially

described by

$$\theta = (A + Bt)\, e^{-\left(\frac{k}{2}\right)t} \tag{15.26}$$

where the constants A and B depend on the strength of the damping.[5]

Problems

53. Determine the period of a simple pendulum with a length of 200 cm and determine the length of a pendulum necessary to have a period of 1.00 second. ans. 2.84 seconds, 24.87 cm

54. Determine the period of a pendulum of length 24.87 cm when connected to the roof of an elevator that is accelerating upward with an acceleration of 1.00 m/sec, a downward acceleration of 1.00 m/sec and in free fall. ans. 0.953 sec, 1.055 sec, ∞

55. A circular disk of mass 2.00 kg, inside radius of 0.60 m and outside radius of 1.00 m is hung from a pivot. Calculate the moment of inertia of the disk about its central axis and then about the pivot point on the inside radius. Then calculate the period of the pendulum. ans. 1.36 $kg - m^2$, 2.08 $kg - m^2$, 2.64 sec

56. A U-tube open at both ends contains Mercury and has an inside cross sectional area of A. The Mercury column has a length of L and is displaced h cm from its equilibrium position when a suction hose is attached to one end. Calculate the frequency and period of vibration after the suction hose is disconnected. ans. $T\frac{1}{2\pi}\sqrt{\frac{2g}{L}}$.

[5] R. A. Becker "Introduction to theoretical Mechanics" McGraw-Hill Book Company, Inc. New York, (1954) pp. 139-146

Chapter 4

STRENGTH OF MATERIALS

In the preceding chapters, we have examined the effect of forces on objects as rigid bodies without consideration of the internal effects of those forces. In this chapter, we will examine the internal effect of forces exerted on materials and the impact of these effects on engineering structures. Bodies will no longer be considered rigid and we will examine the deformation of the bodies.

16 Materials classification and properties

16.1 Classifications

Materials used in construction and fabrication can be divided into four classes, each of which has properties unique to that class of material. The list in table 4.1 illustrates the type of materials that fall into each class.

16.2 Properties of Materials

Materials may be characterized according to their behavior when forces are applied. For this characterization, the **stress** under which the material is placed is normally defined as

$$\sigma = \frac{F}{A} \tag{16.1}$$

where F is the force applied and A is the cross sectional area to which the force is applied so that stress has units of pressure. As illustrated in figure 4.1 forces may

Material Class	Examples
Metals	Iron, Steel, Copper, Aluminum, Titanium and their alloys
Polymers	Polyethylene, Polymethylmethacrylate, Nylon, Polystyrene, Polyurethane, Polyvinylchloride and Rubbers
Ceramics	Crystaline inorganic non-metals such as Alumina, Magnesia, Silicon Carbide, Silicon nitride; non-crystaline solids such as glasses and silicates; cement and concrete
Composites	Wood, Fiberglass, Carbon fiber reinforced polymers, Filled polymers and Cermets

Table 4.1: Classifications of materials

be resolved into components parallel F_t or transverse F_s to the axis of the material being tested leading to the definition of **tensile stress** σ_t or **shear stress** σ_s so that

$$\sigma_t = \frac{F_t}{A} \tag{16.2}$$

$$\sigma_s = \frac{F_s}{A} \tag{16.3}$$

The effect of the applied force on the material is called **strain**, a unitless quanity described in terms of the elongation or lateral displacement of the material. In the case of tensile forces, the strain is defined by

$$\varepsilon = \frac{\Delta L}{L} \tag{16.4}$$

16.3 Stress-Strain Curves

Different types of materials behave differently under the effect of an externally applied stress. In general, the effect of stress on materials can be represented by a curve similar to that illustrated in figure 4.2

In this diagram, point A represents the **proportional limit** or that limit of stress at which strain is directly proportional to stress, point B represents the **elastic limit** or the stress at which the strain will return to zero when the stress is removed, point C represents the **yield point** or that stress at which there is an increase in strain with only an infinitesimal increase in stress, point D represents the **ultimate strength** or the maximum stress needed to produce further strain and point E represents the **rupture strength** or that point at which the material breaks.

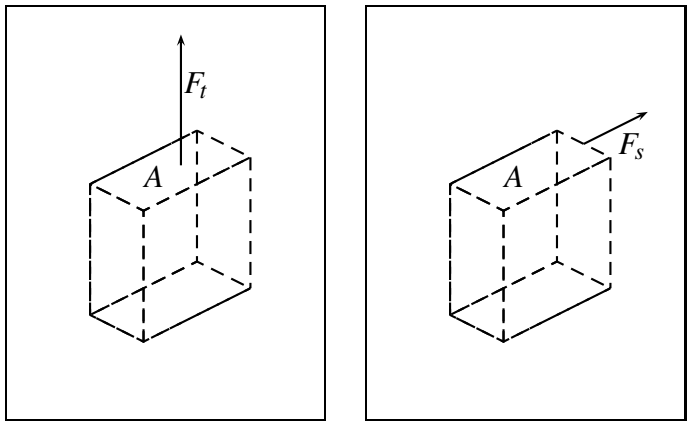


Figure 4.1: Tensile and Shearing Forces

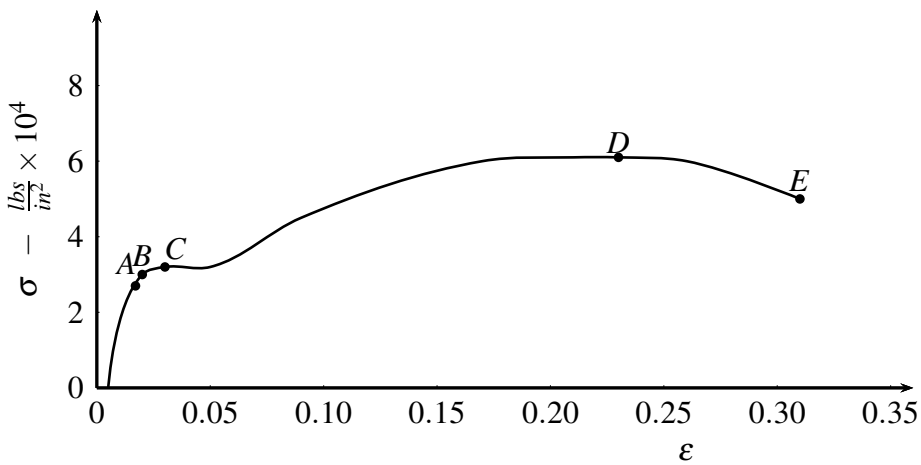


Figure 4.2: Stress-strain curve for typical material

Materials differ greatly in the exact profiles of their curves. For example, metallic materials are commonly classified as **ductile** or **brittle** materials. Brittle materials rupture at a much smaller strain than ductile materials. An arbitrary dividing line between these types of materials is usually taken as a strain value of 0.05.

17 Hooke's law

There are four ways to put materials under stress: (a) linear stretching and compression, (b) shearing stress, (c) volumetric stress and (d) twisting or torsional

stress. Hooke's law may be applied to all of these to obtain mathematical relations between stress and strain.

17.1 Linear Stress

In the proportional range of the stress-strain curve, there is a linear relation between the tensile stress and tensile strain. This proportionality was first noticed by Sir Robert Hooke in 1678 and still bears his name.

$$\sigma = Y\varepsilon \tag{17.1}$$

The constant of proportionality is often called **Young's Modulus** after Thomas Young who devised experiments to measure the constant of proportionality. Young's modulus has units of pressure and is normally the same for compression and stretching of materials.

Example

Consider a rod made from rolled copper which has a Young's modulus of 4.68×10^7 psi and a diameter of $\frac{1}{4}$-inch and area of $0.196\ in^2$. What will the strain be if the rod supports a weight of 1000 lb?

$$\sigma = \frac{F}{A} = \frac{1000}{0.196} = 5093\ psi \tag{17.2}$$

$$\frac{\Delta L}{L} = \frac{\sigma}{Y} = \frac{5093}{4.68 \times 10^7} = 0.000109,\ or\ 0.0109\% \tag{17.3}$$

17.2 Shearing Stress

As mentioned earlier, applied forces can also produce a shearing stress in materials. Hooke's law is also applied in this case, but the method of calculating the shearing strain is different, as illustrated in figure 4.3

In this case, the shearing stress is calculated from the same formula as tensile stress, but the shearing strain is calculated from the relation.

$$\varepsilon = \frac{\Delta x}{h} \tag{17.4}$$

Then applying Hooke's law leads us to the relation between shearing stress σ and shearing strainε with the constant of proportionality S defined as the **Shear Modulus**

$$\sigma = S\varepsilon \tag{17.5}$$

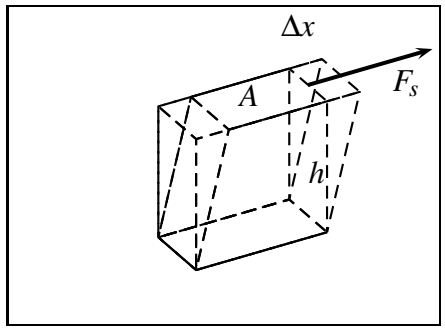


Figure 4.3: Shearing Forces

Example

As an example, suppose that a bolt of 0.25-inch diameter and area 0.049 in^2 is made from a high strength steel alloy $(ASTM\ A514)$ which has a yield strength of $100,072$ psi and a shear modulus is $S = 1.16 \times 10^7$ psi. Taking the yield strength as an approximation of the stress before the bolt undergoes permanent deformation σ_Y allows us to estimate the strain the bolt will experience at that load.

$$\frac{\Delta x}{h} = \frac{\sigma_Y}{S} = \frac{1.00 \times 10^5}{1.16 \times 10^7} = 0.00862 \tag{17.6}$$

and the load that will cause permanent deformation when placed on the bolt is

$$F = \sigma_Y A = 1.00 \times 10^5 \times 0.049 = 4900\ lb \tag{17.7}$$

17.3 Volumetric stress

Another common way in which stress is applied to materials occurs when the force is applied from all directions either as a compression as in the case of an object submerged below water or as expansion as in the case of a balloon rising above the earth into less dense air. This leads to a similar definition of bulk stress, bulk strain and the relation between bulk stress and bulk strain.

$$\sigma = \frac{F}{A} \tag{17.8a}$$

$$\varepsilon = \frac{\Delta V}{V} \tag{17.8b}$$

$$\sigma = B\varepsilon \tag{17.8c}$$

The constant of proportionality is known as the **Bulk Modulus**.

Example

As an example, the water pressure at $5,000$ feet below sea level is $2167\ lb/in^2$ and the bulk modulus of water is 3.19×10^5 psi. Therefore, the percentage decrease in the volume of water at this depth is

$$\frac{\Delta V}{V} = \frac{2,167}{3.19 \times 10^5} = 0.68\ \% \tag{17.9}$$

while the percentage decrease in volume for a solid steel ball with a bulk modulus of 23×10^6 psi is considerably less.

$$\frac{\Delta V}{V} = \frac{2167}{23 \times 10^6} = 0.0094\ \% \tag{17.10}$$

17.4 Rotational stress

Consider the solid cylinder of radius r and length L rigidly clamped at one end and subjected to a twisting torque τ at the other end. We can define the rotational shearing or twisting strain ε_{rs} as the angle of twist and calculate it from the formula

$$\varepsilon_{rs} = \frac{r\theta}{L}, \tag{17.11}$$

where the quantity θ/L is the angle of twist per unit length. To compute the torsional stress it is convenient to recall equation 13.36, the defining equation for the mass moment of inertia, and define by analogy the **area moment of inertia** or the **centroid of area** as[1]

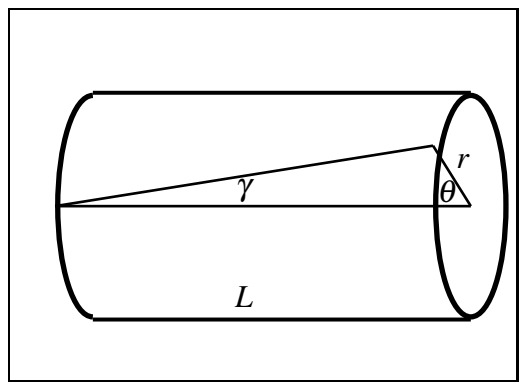


Figure 4.4: Cylinder under rotational stress.

$$I_A = \int_0^r t^2 2\pi t dt = \frac{1}{2}\pi r^4. \tag{17.12}$$

We next define the rotational stress or twisting stress, σ_{rs}, as the applied torque multiplied by the radius and divided by the area moment of inertia to get

[1]The centroid of area corresponds to the center of gravity of a plate of infinitesimal thickness and is often misnamed as center of gravity.

$$\sigma_{rs} = \frac{r\tau}{I_A} \tag{17.13}$$

First we should note that both the twisting stress and the twisting strain are proportional to the distance from the central axis of rotation with their maximum value at the surface. Then applying Hooke's law, $\sigma_{rs} = S\varepsilon_{rs}$, for the proportionality region to relate the twisting stress to the twisting strain we obtain[2]

$$\frac{r\tau}{I_A} = S\frac{r\theta}{L}, \tag{17.14}$$

where S is the shearing modulus. This equation can be used to calculate the angle through which the cylinder is twisted as a result of the applied torque.

$$\theta = \frac{L\tau}{SI_A} \tag{17.15}$$

Thinking of the cylinder as a transmission shaft, we can further relate the angle of twist to the work output of the engine working to turn the cylinder by noting that 1 HP = $33,000$ lb-ft/min and using equation 13.53 to obtain a relation between the power output of the engine, or power transmitted by the shaft, to be

$$\tau = \frac{33,000 \times HP}{\omega}\, ft - lb \tag{17.16}$$

where HP is the power output of the engine and ω is the rpm or revolutions per minute of the shaft.

Example

As an example, calculate the angle of twist in a steel shaft with a shear stress modulus of $1.2 \times 10^7\ lb/in^2$, a diameter of $1\frac{3}{4}$ inches and a length of 48 inches that experiences a torque of $10,000$ lb-in. For this purpose, we note that the area moment of the shaft

$$I_A = \frac{1}{2}\pi r^4 = \frac{1}{2}(3.1416)(0.875)^4 = 0.92\ in^4 \tag{17.17}$$

[2]Other terms such as shear stress and shear strain are more common definitions in engineering texts, but here we use the word twisting stress to distinguish it from earlier use of shearing stress when the bolt is severed by a perpendicular force.

From this the torsional stress can be calculated

$$\sigma = \frac{r\tau}{I_A} = \frac{(0.875)(10,000)}{0.9208} = 9503\ lb/in^2 \tag{17.18}$$

and the angle of twist can be obtained from equation 17.15

$$\theta = \frac{L\tau}{SI_A} = \frac{(48)(10,000)}{(1.2\times 10^7)(0.9208)} = 0.0435\ radians = 2.49^o \tag{17.19}$$

The HP of an engine needed to drive the shaft with this torque, $10,000\ lb-in = 833.33\ lb-ft$, and at 2000 rpm is then obtained from equation 17.16 after converting from revolutions per minute to radians per minute.[3]

$$HP = \frac{\tau\omega}{33,000} = \frac{833.33\times 2000\times 2\pi}{33000} = 317\ HP \tag{17.20}$$

It is interesting to compare this example to the power train of an automobile and it may be surprising to find out the stress placed on drive shafts. It should also be noted that a hollow cylinder for a drive shaft would have twice the area moment and thus the torque would be one-half as great as in this example.

17.5 Variation with temperature

The modulus of elasticity of most materials decreases with an increase in temperature. As an example, the temperature dependence of Carbon steel is displayed in figure 4.5

Problems

57. Calculate the load a 0.440-inch diameter steel cable can safely carry before reaching the yielding stress and the fractional elongation at that load. Take the yield strength to be 82.0 MPa and Young's modulus to be 211 GPa. ans. 1807 lb and 0.039%.

58. A steel rod of cross-sectional area 2.50 in^2 protrudes from a rigid mount. A tractor hitched to the rod with a cable attached close to the mount exerts a pull of 2000 lb perpendicular to the axis of the bolt. Take S = 82.0 GPa What is the shearing stress and the shearing strain developed in the rod? ans. 800 psi and 0.00689%

[3] 1HP=33,000 $\frac{ft\cdot lb}{min}$; 1rpm = $\frac{1}{60}$ rps = $\frac{2\pi}{60}$ rad/sec

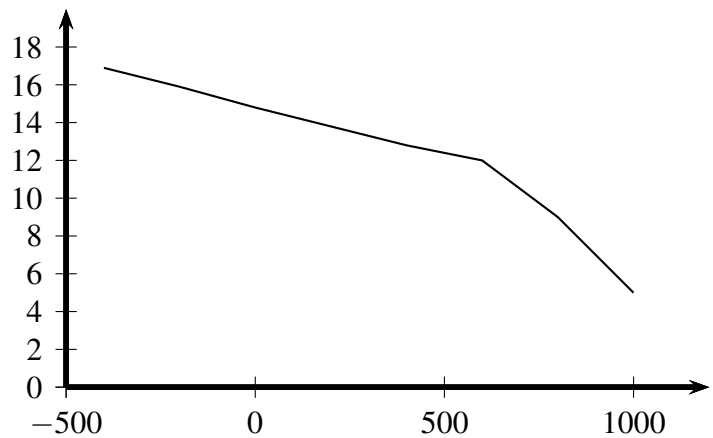


Figure 4.5: Temperature dependence of modulus of elasticity for Carbon Steel.

59. A flexible container is filled with ethyl alcohol, which has a bulk modulus of 1.305×10^5 psi and lowered to a depth of 1000 feet in the ocean. What is the percentage decrease in volume? ans. 0.332%

60. A mechanic attaches a wrench with a 2.00-ft long handle to a stubborn head bolt in an engine block and exerts a pull of 100 lb at the end of the handle. The head bolt has a diameter of 0.500 inches and the threads begin 2.00 inches below the bolt head. What is the torsional stress, the torsional strain and the angle of twist? Does the bolt yield if it is made from high strength ASTM A514 steel and has a yield strength of 690 MPa and shear modulus of 76 GPa? ans. 97,897 psi, 0.88%, 4.07^o and almost.

18 Applications

18.1 Thin-Walled Cylinders

Consider a thin-walled cylinder closed at both ends in which compressed air is stored. Let the cylinder have a length L, a wall thickness h and an inner radius r as illustrated in figure 4.6.

The lines representing the force of the compressed gas against the cylinder wall radiate outward from the center axis of the cylinder. The horizontal components of these force vectors cancel leaving only the vertical components to reinforce. The accumulated total of these forces will tend to separate the cylinder at

the halfway points. The total force upward is obtained from

$$\int_0^{\pi} P(rd\theta)(\sin\theta)L = 2PrL \tag{18.1}$$

The tangential stress induced in the cylinder wall needed to provide a counterbalancing force can then be obtained from

$$2hL\sigma = 2Prl \tag{18.2}$$

A factor of 2 is included since there are two sides to the cylinder. This results in a value for the stress induced in the cylinder walls

$$\sigma = \frac{Pr}{h} \tag{18.3}$$

Since the ends of the cylinder are flat, the longitudinal force tending to separate the ends from the cylinder is simply $P\pi r^2$ and the opposing force needed to counterbalance this force $2\sigma_L \pi rh$ show the longitudinal stress σ_L to be one-half that of the tangential stress

[h!]

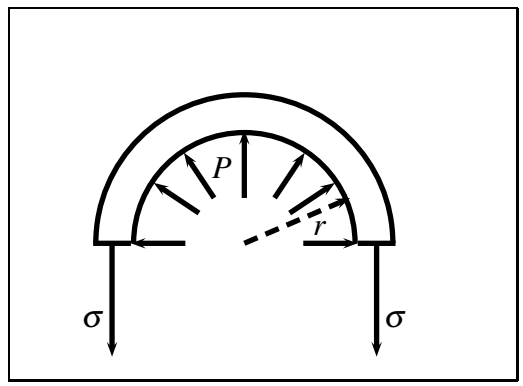


Figure 4.6: Cross sectional view of a cylinder under torsional stress.

$$\sigma_L = \frac{Pr}{2h} \tag{18.4}$$

18.2 Thin-Walled Spheres

If the vessel is a sphere instead of a cylinder and subjected to a pressure P, the force tending to separate two halves of the sphere at an arbitrary dividing line will be

$$F = \int_0^{\pi/2} (P\sin\theta)(2\pi r\cos\theta) rd\theta \tag{18.5}$$

$$= 2\pi Pr^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta \tag{18.6}$$

$$= \pi r^2 P \tag{18.7}$$

As a result of this force, the force in the walls of the sphere necessary to resist the separating or crushing force will be

$$2\pi rh\sigma = \pi r^2 P \tag{18.8}$$

and the stress developed in the walls of the vessel will be

$$\sigma = \frac{Pr}{2h} \tag{18.9}$$

which is one-half of the stress developed in cylinder walls.

Example

The Trieste, designed by the Swiss scientist Auguste Piccard and originally built in Italy, was later fitted with a new pressure sphere manufactured by the Krupp Steel Works of Essen, Germany, with a diameter of approximately 7 feet and a hull thickness of 5 inches. It successfully reached a depth of $35,813$ feet January 23, 1960 in the Marianas Trench. The steel use was HY-100 with an elastic coefficient of 2.97×10^7 psi and yield strength of 9.993×10^5 psi. Calculate the shrinkage of the circumference and diameter of the pressure hull at that depth.

At a depth of $35,813$ feet, the pressure from the weight of the sea water above the diving bell is

$$P = \rho gh = \frac{(62.4\ psf)(35,813\ ft)}{144\ sq.in/sq.ft)} = 1.55 \times 10^4\ psi. \tag{18.10}$$

The stress induced in the hull can be calculated from equation 18.9

$$\sigma = \frac{Pr}{2h} = \frac{(1.55 \times 10^4\ psi)(42\ in)}{2(5\ in)} = 6.50 \times 10^4\ psi. \tag{18.11}$$

So that the fractional compression around the circumference of the hull would be

$$\frac{\Delta L}{L} = \frac{\sigma}{Y} = \frac{6.5 \times 10^4}{2.97 \times 10^7\ psi} = 0.00218\ or\ 0.218\%, \tag{18.12}$$

which would be the fractional shrinkage of the circumference and diameter of the diving bell at that depth.

Problems

61. A steam locomotive developed pressures of 500 psi in a boiler having a diameter of 3 feet and length of 5 ft. What is the minimum thickness of steel that would be required to contain the pressure with a safety factor of 3? Assume that the yield strength of the steel is 90 MPa. ans. 2.07 in

62. A spherical satellite is sent into space with an inert gas contained inside at a pressure of 1 atm. If the diameter of the probe is 18 inches, what is the minimum wall thickness needed to contain the pressure assuming a yield strength of 90 MPa? ans. 5.1 mils

63. A steel ball of radius 1 meter and wall thickness 1 cm is pulled down with a heavy weight into the deepest part of the Ocean. If the yield strength of the steel is 6.895 $\times 10^8$ Pa, at what depth will the ball implode? ans. 2811 m

19 Thermal Expansion of Materials

All materials undergo changes in their parameters as their temperature changes. Normally, material parameters increase with temperature but in some instances, notably water, volume increases near critical points. The general formula for representing changes in material parameters with temperature can be written as

$$\frac{d\chi}{dT} = \alpha\chi, \tag{19.1}$$

where χ is the parameter under study. Three parameters are normally of concern, length, area and volume in which case we often write

$$\left(\frac{dL}{dT}\right) = \alpha_L L \qquad for\ length \tag{19.2}$$

$$\left(\frac{dA}{dT}\right) = \alpha_A A \qquad for\ area \tag{19.3}$$

$$\left(\frac{dV}{dT}\right) = \beta V \qquad for\ volume \tag{19.4}$$

For **isotropic materials** that have equal **temperature expansion coefficients** for each axis of expansion, the coefficients for expansion can be related. In the case

of area expansion, we can start with $A_o = L_o^2$ and after expansion $A = L^2$ with $L = L_o + \Delta L$

$$\frac{\Delta A}{A_o} = \frac{\left(L_o^2 + 2L_o\Delta L + (\Delta L)^3\right)}{L_o^2 - L_o^2} =\simeq 2\frac{\Delta L}{L_o} \tag{19.5}$$

$$\alpha_A \Delta T \simeq 2\alpha_L \Delta T \quad so\ that \tag{19.6}$$

$$\alpha_A \simeq 2\alpha_L \tag{19.7}$$

By a similar process

$$\beta \simeq 3\alpha_L \tag{19.8}$$

For **anisotropic materials** that have different expansion coefficients along each axis, these approximations do not work. In general, the thermal expansion coefficients for most solids are relatively constant over a wide range of temperature. However, a notable and important exception exists in the case of water.

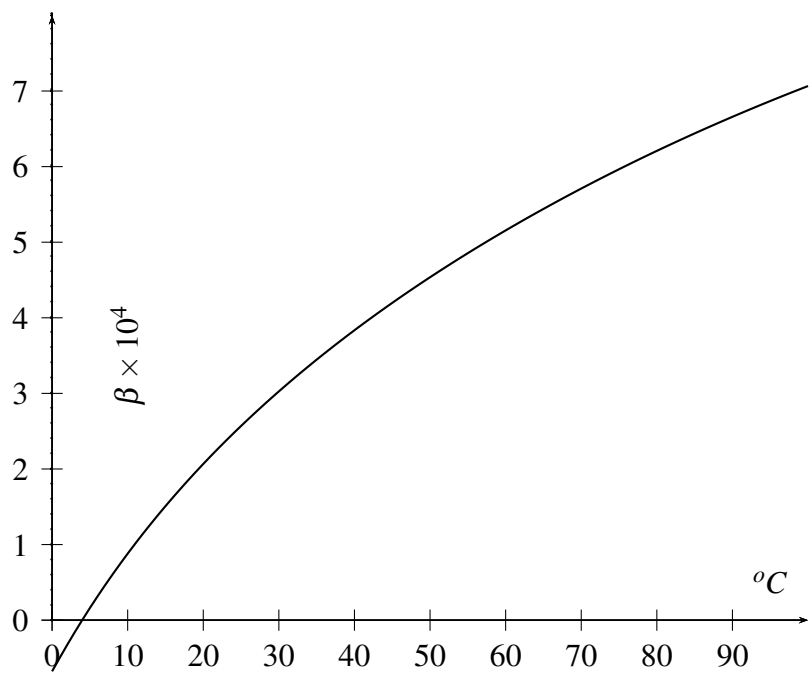


Figure 4.7: Volume expansion coefficient for water.

As may be noted in figure 4.7, the coefficient of expansion for water is negative below about 4 ^oC. As a result, the density of water has a maximum near 4 ^oC. This is why ice floats and why the temperature of water below the surface may

be 4 ^oC while the surface temperature of water is 0 ^oC or below. The **thermal expansion coefficients** for the metallic elements are listed in the appendices.

Numerous applications are common for the expansion of metals with an increase in temperature. It may be noted that the diameter of a hole will increase as the metal is heated. This makes possible "shrink fitting" of bushings on metal shafts and in earlier times metal rims on wagon wheels at the blacksmith's forge. One of the more difficult problems in engineering was matching the temperature expansion coefficient of ceramics to that of a metal surface to prevent crinkling. Notable successes are in automotive sparkplugs and tires, cooking ware, adapting engines to operate over a wide range of temperatures, in civil engineering work and many others. One other notable evidence of the effect of temperature expansion is in the expansion of railroad irons and in bridge structures.

Example

The George Washington Bridge across the Hudson river is 1 mile long. Take the thermal expansion coefficient as 12.2×10^{-6} and calculate the approximate change in length for a temperature increase of 30 ^oC.

$$\Delta L = \alpha L \Delta T = (12.2 \times 10^{-6})(5280)(30) = 1.93\ ft. \tag{19.9}$$

As another example, a steel rod 1 meter long on earth will shrink if it is taken to one of the earth satellites where the temperature is near absolute zero by

$$\Delta L = \alpha L \Delta T = (12.2 \times 10^{-6})(100)(293) = 0.36\ cm. \tag{19.10}$$

Problems

64. A steel bar is placed between two rigid blocks and the temperature raised by heating 100 ^oC. What is the stress developed in the bar to prevent elongation? Take α=1.22 $\times 10^{-5}$ $/^oC$ and Y = 211 GPa. ans. 37,330 psi

65. A steel rod has a diameter of 1.000 cm when measured at 20 degrees C. What will the diameter be when heated to 250 degrees C. Take $\alpha = 1.22 \times 10^{-3}$ deg$C$. ans. 1.003 cm

66. Calculate the change in volume of a copper sphere of radius 10 cm when heated from 20 degrees C to 420 degrees C. Take $\alpha = 1.60 \times 10^{-5}$ deg$C$. ans. 80.4 cm^3

67. A copper ring has an inside diameter of 4.980 cm at 20.0 degrees C. To what temperature must it be heated to fit over a shaft 5.000 cm in diameter? ans. 270 degrees C.

68. A clock pendulum with a brass has a period of exactly 1.00 seconds. How many seconds will it loose in 24 hours if the temperature rises 40.0 degrees C. Take $\alpha = 1.80 \times 10^{-5}$ deg$C$. ans. 31 seconds

20 Structures in equilibrium

A building truss is a static structure designed so that the stress resulting from loads will be distributed as tension along one of the members of the truss producing either elongation or compression of that truss member. to calculate the forces acting on a truss, we can use basic principle that for any static structure in equilibrium, the sum of all the torques about any given point must be zero, and that the sum of all the components of all forces acting at any one point must be zero.

$$\sum \tau_i = 0 \tag{20.1}$$
$$\sum F_x = 0 \tag{20.2}$$
$$\sum F_y = 0 \tag{20.3}$$

20.1 Pratt bridge truss

As an example, consider the Pratt truss depicted in figure 4.8.[4] In this truss, a weight of 5,010 lb is distributed between the center two sections, and the truss is supported at each end by a supporting block. The length of the truss is 60 feet equally distributed between the 6 sections. The height is 7.5 feet and the truss is constructed on a $3-4-5$ triangle making each angle 36.87 degrees. The load at each end is determined by requiring the sum of all the toques to be zero. Consider first the torques about the point L.

$$60F - 40(2000) - 30(2000) - 20(2000) = 0 \tag{20.4}$$
$$F = 3000\,lb \tag{20.5}$$

The load at the opposite end will be the same due to symmetry of the loading. Consider next the point A and require the sum of all the forces to be zero. The

[4]The Pratt truss was invented in 1844 by Thomas and Caleb Pratt.

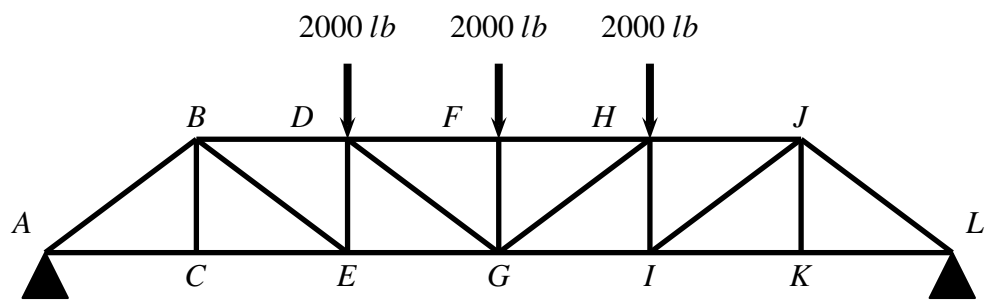


Figure 4.8: Pratt Bridge truss

force in all members of the truss will be a force of tension or a force of compression. If a mistake is made defining whether the force in any member is one of tension or compression, the answer will come out with a negative sign indicating that the opposite choice should have been made. For the y-components of the forces, we have

$$-F_{AB}\sin 36.87 + 3000 = 0 \tag{20.6}$$

$$F_{AB} = 5000\,C\,lb. \tag{20.7}$$

This force is labeled with a C to denote compression. For the x-components of the forces, it is clear that the horizontal truss member must be under tension to counter the horizontal force exerted by F_{AB}.

$$-5000\cos 36.87 + F_{AC} = 0 \tag{20.8}$$

$$F_{AC} = 4000\,C\,lb. \tag{20.9}$$

This process can be continued from one joint to another always picking the next joint so that the number of unknown forces does not exceed the number of equations that can be solved simultaneously. It is helpful, if not necessary, to draw force diagram for each joint. It should be noted that the force in the vertical members between B and C and also J and K must be zero since there is no vertical

opposing force. The results of the analysis may be summarized as follows:

$$F_{AB} = F_{JL} = 5000\,T \tag{20.10}$$
$$F_{BE} = F_{JI} = 5000\,C \tag{20.11}$$
$$F_{DG} = F_{HG} = 1667\,C \tag{20.12}$$
$$F_{BD} = F_{HJ} = 8000\,T \tag{20.13}$$
$$F_{FD} = F_{FH} = 9333\,T \tag{20.14}$$
$$F_{AC} = F_{CE} = F_{IK} = F_{KL} = 4000\,C \tag{20.15}$$
$$F_{EG} = F_{GI} = 8000\,C \tag{20.16}$$

From these results, the cross sectional area of the truss members and the modulus of elasticity, the fractional elongation or compression can be determined. Since all the sections are triangles, the structure is internally stable and the load bearing capacity is determined by the yield strength of the members and in particular the yield strength of the two upper center members.

Problems

69. Determine the tensions in each member of the Howe bridge truss shown in figure 4.9 taking the triangles to be $3-4-5$ triangles. ans. $F_{AB} = 500$, $F_{AC} = 400$, $F_{BC} = 300$, $F_{BD} = 400$, $F_{CD} = 500$, $F_{CE} = 800$, $F_{BD} = 400$, $F_{DF} = 800$, $F_{DE} = 100$, $F_{EF} = 187$, $F_{EG} = 933$, $F_{FG} = 0$.

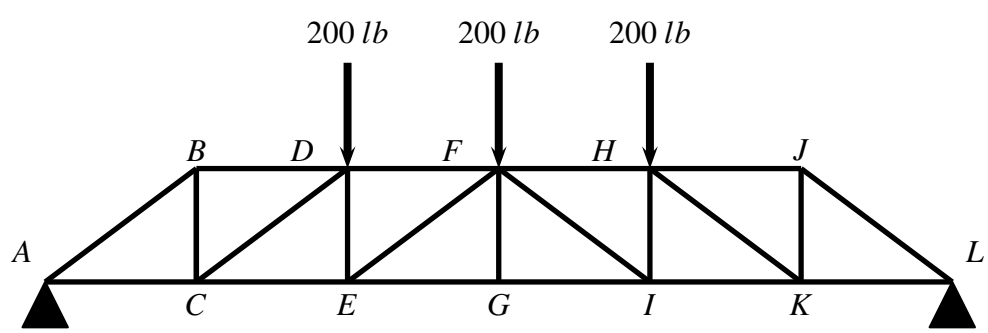


Figure 4.9: Howe truss

70. Find the force in each member of the cantilever truss illustrated in figure 4.10. ans. AB 100 lb, BC 100 lb, CD 87 lb, DE 116 lb, BD 0 lb, AD 58 lb

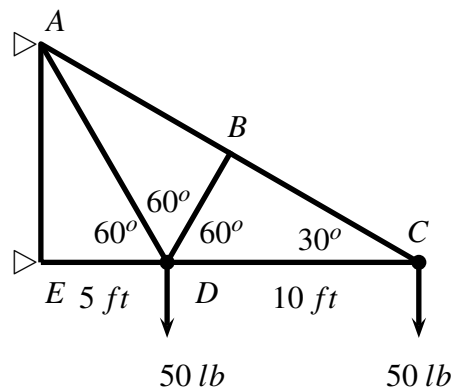


Figure 4.10: Cantilever truss

Chapter 5

FLUID MECHANICS

Several definitions of solids, fluids and gases have been offered over the years, but none are completely rigorous to the exclusion of other definitions. However, a satisfactory means of distinguishing fluids from solids is to define a fluid as a substance that will deform, or flow, under much smaller shearing forces than solids. In addition to that distinction, it may be noted that the intermolecular spacing of both fluids and solids is essentially constant while the intermolecular spacing of molecules in gases can vary over a wide range. It may also be noted that the molecules of a solid are arranged in a fixed lattice structure whereas the molecules of a fluid can vary their arrangement without restriction. Internal forces tend to restore solids to their original shape after a deforming force is removed, but fluids tend to remain the shape achieved under the force of deformation.

A fluid may be either a liquid or gas. To distinguish between fluids and gases, it may be said that a liquid tends to be incompressible while the volume of gases is sensitive to even small changes in pressure or temperature. The discussion in the chapter on materials was restricted to solids, but this discussion will encompass both liquids and gases since similar physical concepts apply.

On a relevant historical note, it should be noted that the study of fluid mechanics undoubtedly began in the middle eastern civilization several thousand years ago, but the first recorded attempt to define the physical principles of fluid dynamics and fluid statics did not begin until about 250 B.C. with the effort of Archimedes. The fundamental principles of classical hydrodynamics were developed by Newton, Bernoulli, Euler and others in the seventeenth and eighteenth centuries. This development was continued in the nineteenth century by Reynolds, Prandtl, Riemann, Mach and others many of whom are remembered by the names of laws and principles they discovered. Several physical variables that find com-

mon use in the study of fluids are identified in table 5.1.

Variable	Definition	Units
Pressure	$p = \frac{F}{A}$	$Newtons/meter^2$
Density	$\rho = \frac{M}{V}$	$kilograms/meter^3$
Volume flow rate	$Q = \frac{V}{A}$	$meter^3/second$
Mass flow rate	$W = Q\rho$	$kilograms/sec$
Viscosity	$\mu = T\frac{dV}{dy}$	$Poise$
Surface Tension	$\gamma = \frac{dF}{dA}$	$Newtons/meter$
Specific gravity	$s = \frac{\rho}{\rho_{water}}$	

Table 5.1: Physical terms used in fluid mechanics.

Physical characteristics of fluids are also defined and listed in table 5.2. These characteristics may be applied to both liquids and gases.

1. Fluid flow may be steady or non-steady.
2. Fluid flow may be rotational or non-rotational.
3. Fluid flow may be compressible or non-compressible.
4. Fluid flow may be viscous or non-viscous.
5. Fluid flow may be laminar or turbulent.

Table 5.2: Characteristics of fluid flow.

21 Ideal Gases

We can commence a discussion of fluids by defining an **ideal gas** as one which consists of a large number of identical molecules all in independent random motion and require that no forces act on the molecules except during collisions with other molecules and that all collisions are elastic and of short duration.

21.1 Kinetic Theory

For such an ideal gas, the force on the containment wall will be the time rate of change of the momentum as the molecules collide with the containment. For simplicity, we will assume the containment is a cube of side L which contains a total of N molecules each of mass m. Resolving the velocity vector of each molecule into perpendicular components V_x, V_y and V_z, the change in momentum for any one collision will be simply $2mV_i$ where i refers to the component of velocity perpendicular to that wall. The time for the molecule to travel across a container a length L will be L/V_i and twice that for a round trip so that the number of collisions with the wall per unit time for that one molecule will be $V_i/2L$ and the rate at which that molecule transfers momentum to the wall is $2mV_i \times V_i/2L$ = mV_i^2/L. Then the pressure exerted on the wall is the sum of the momentum imparted by all the molecules divided by the area of the wall L^2.

$$P = \frac{1}{L^2}\sum_1^N \frac{mV_i^2}{L} \tag{21.1}$$

$$= \frac{mN}{L^3}\left[\sum_1^N \frac{V_i^2}{N}\right] \tag{21.2}$$

The quantity in brackets is the average value of the molecular velocity perpendicular to one wall, which is $1/3$ the average value of the total velocity of each molecule since there are three components of the velocity, $V^2 = V_x^2 + V_y^2 + V_z^2$, and the average values of each component are equal. Therefore

$$P = \frac{1}{3}\rho\bar{V^2}, \tag{21.3}$$

where $\rho = mN/L^3$ is the density of the gas. We can therefore define the **root mean square** velocity of each molecule in terms of the pressure exerted on the walls of the containment and the density of the gas.

$$V_{rms} = \sqrt{\frac{3P}{\rho}} \tag{21.4}$$

At this point it is convenient to introduce **Avagadro's hypothesis**. Avagadro's hypothesis named for Amedeo Avagadro (1776-1856) an Italian Physicist and mathematician who first published the hypothesis in 1811, states that *all gases*

occupying the same volume at the same temperature and pressure contain equal numbers of molecules. The experimentally determined value of this number is

$$N_o = 6.02214179(30) \times 10^{23} mol^{-1} \tag{21.5}$$

Using Avagadro's number and noting that $L^3 = V$, the volume of the gas, we can rewrite equation 21.2 as

$$PV = \frac{1}{3} mnN_o \bar{V^2}, \tag{21.6}$$

where n is the number of moles of the gas. However the average translational energy, or kinetic energy, per molecule of the gas can be written as

$$K = \frac{1}{2} m\bar{V^2}, \tag{21.7}$$

so that equation 21.6 can now be combined with equation 21.7 to obtain

$$PV = \frac{2}{3} nN_o K. \tag{21.8}$$

By these formulas, we have expressed the product of the pressure and volume in terms of microscopic variables of molecular velocity and kinetic energy. These microscopic variables are, however, hard to observe and measure.

21.2 Temperature of an Ideal Gas

A macroscopic expression of the pressure and volume can begin with the definition of the temperature as a macroscopic variable that can be measured using physical properties of materials. For this purpose, we define the **temperature** of an ideal gas as proportional to the average translational kinetic energy of the molecules

$$T = (constant) \times \frac{1}{2} m\bar{V^2} \tag{21.9}$$

Combining this definition with equation 21.3, we see that the pressure of the ideal gas is proportional to the temperature of the ideal gas.

$$P = (constant)T \tag{21.10}$$

From this equation, it is apparent that at least one thermometric quantity, P can act as a barometer for measurement of temperature. In this case the temperature would ideally be zero when the pressure and the average translational kinetic energy of the molecules is zero. The point at which there is no translational motion of the molecules is known as **absolute zero**.

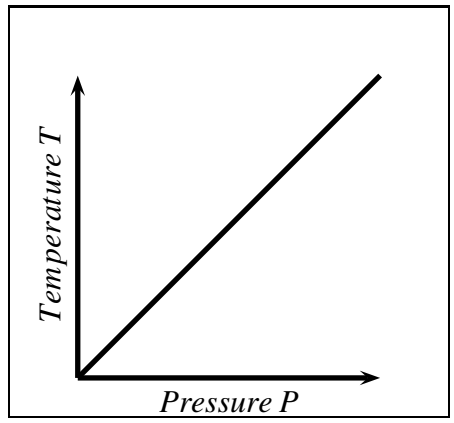


Figure 5.1: Cylinder under torsional stress.

21.3 Temperature Scales

The slope of the line and temperature scale has been defined in several ways over the last three centuries. In addition to the zero value the **triple point** of water, at which water, ice and water vapor coexist in equilibrium, the **steam point** of water, at which steam and liquid water are in equilibrium and the **ice point** of water, at which ice and air-saturated water coexist in equilibrium, have served as indices. For the centigrade scale, named after the Swedish astronomer Anders Celsius (1701-1744), the ice point is taken as zero and the steam point is taken as 100 degrees. For the Fahrenheit scale, named after, the physicist Daniel Gabriel Fahrenheit (1686-1736), the ice point is also taken as zero but the steam point is taken as 212 degrees. For the Rankine scale, named after the Scottish engineer and physicist William John Macquorn Rankine, who proposed it in 1859, the zero point is taken at absolute zero but the ice point of water is taken as 491.67 degrees. These scales are summarized below and give rise to conversion formulas.

Calibration Point	Kelvin	Celsius	Fahrenheit	Rankine
absolute zero	0	-273.15	-459.67	0
ice point	273.15	0	32	491.67
triple point	273.16	0.1	32.018	491.688
steam point	373.1339	99.9839	211.9710	671.641

Table 5.3: Calibration points for temperature scales.

$$1^oC = 1.8^oF = 1.8^oR \tag{21.11}$$
$$^oF = {}^oC \times \frac{9}{5} + 32 \tag{21.12}$$
$$^oK = {}^oC + 273.15 \tag{21.13}$$
$$^oR = {}^oC + 273.15 \tag{21.14}$$

21.4 Ideal Gas Laws

Before defining what today is called the **ideal gas law** it is useful to examine some of the historical gas laws that were published during the 17th, 18th and 19th centuries.[1] **Boyle's law**, named after chemist and physicist Robert Boyle (1627-1691) who published the original law in 1662, states that at constant temperature the volume of a gas decreases as pressure is applied.

$$V = \frac{k_1}{P} \tag{21.15}$$

Charles law, investigated the French Scientist, inventor and mathematician Jacques Charles (1746-1823) and published by Gay-Lussac in 1802 states that at constant pressure the volume of a gas increases with temperature.

$$V = k_2 T \tag{21.16}$$

Gay-Lussac's law, investigated by the French Chemist and Physicist Joseph Louis Gay-Lussac ((1778-1850) and published in 1802, states that at constant volume the pressure of a gas is proportional to the temperature.

$$P = k_3 T \tag{21.17}$$

This law should probably be attributed to Guillaume Amontons (1663-1705) a French physicist and instrument maker who originally discovered the law between 1700 and 1702. These three laws are often combined to form what became known as the **general gas law**.

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \tag{21.18}$$

[1] Considerable discussion has taken place regarding who should be given credit for which law. The credits cited in this text are not intended to be exact but credit must be given to all the scientists named above who worked to define these laws during the early stages of science

It is clear from the foregoing that volume is a function of temperature and pressure $V = V(T,P)$. Differentiating the function $V = V(T,P)$, substituting for $\left(\frac{\partial V}{\partial P}\right)_T$ from Boyle's law and for $\left(\frac{\partial V}{\partial T}\right)_p$ from Charles' law, dividing by V and integrating gives

$$dV = \left(\frac{\partial V}{\partial P}\right)_T dp + \left(\frac{\partial V}{\partial T}\right)_P dT \tag{21.19}$$

$$\int \frac{dV}{V} = -\int \frac{dP}{P} + \int \frac{dT}{T} \tag{21.20}$$

$$\ln V = -\ln p + \ln T + constant \tag{21.21}$$

Rearranging the latter equation, we see that the product of pressure and volume is proportional to the temperature.

$$pV = (constant)T \tag{21.22}$$

Experimental results have shown that 1 mole of an ideal gas occupies 22.414 liters at 0^oC or 273.16^oK. This allows calculation of the constant

$$R = \frac{pV}{T} = \frac{(1atm)(22.414)}{273.16} = 0.08205 \, \frac{liter-atm}{mole\, ^oK} \tag{21.23}$$

This constant is known as the **molar gas constant**.[2]. Now equation 21.22 can be written as follows where n is the number of moles of the gas.

$$PV = nRT \tag{21.24}$$

Thus we have achieved a formal relation between the macroscopic variables for pressure, volume and temperature. This equation is know as the **ideal gas law** and is the law that we will use throughout the following discussions. Equations such as the ideal gas law that give the relation between pressure, volume and temperature for a substance are often referred to as an **equation of state**.

21.5 Boltzman's Constant

It is tempting to compare equation 21.24 with equation 21.8.

$$PV = \frac{2}{3} n N_o K = nRT \tag{21.25}$$

[2] $R = 0.08205 \, \frac{liter-atm}{deg-mole} = 8.314 \, \frac{joules}{deg-mole} = 1.987 \, \frac{calories}{deg-mole}$; The most accurate value reported by CODATA (2006) is $R = 8.314472(15) J/mol/^oK$

From this comparison, it is seen that we can write the Kinetic energy and the ideal gas law in terms of another constant k.

$$K = \frac{3}{2}kT \tag{21.26}$$

$$PV = nN_o kT \tag{21.27}$$

where the constant is called **Boltzman's constant** and defined by

$$k = \frac{R}{N_o} = 1.3806504(24) \times 10^{-23}\, Joules/molecule/^oK \tag{21.28}$$

and named for Ludwig Boltzman (1844-1906) Austrian scientist and academician. Therefore, it may be said that Boltzman's constant bridges the gap between the microscopic and macroscopic descriptions of an ideal gas. Boltzman's constant plays an important role in several areas of physics and will continue to reappear throughout our studies.

21.6 Dalton's Law

Dalton's law of partial pressures states that the total pressure exerted by a gaseous mixture is equal to the sum of the partial pressures of each individual component in a gas mixture. This empirical law is named after John Dalton (1766-1844) English chemist, metrologist and physicist in 1801. To express this law in mathematical terms, we can think of a container of volume V containing several gases where n_i represents the number of moles of the ith gas and T represents the temperature uniform throughout the container for all gases. Assuming that all gases behave independently of one another, we can say that the total pressure is the sum of the **partial pressures** P_i that would be exerted by each gas if it were alone in the container of volume V at a temperature T and use the general gas law to write:

$$\begin{aligned}
P &= \frac{n_1RT}{V} + \frac{n_2RT}{V} + \frac{n_3RT}{V} + \cdots \\
&= \left(\frac{n_1}{n} + \frac{n_2}{n} + \frac{n_3}{n} + \cdots\right)\frac{nRT}{V} \\
&= \left(\frac{n_1}{n} + \frac{n_2}{n} + \frac{n_3}{n} + \cdots\right)P \\
&= P_1 + P_2 + P_3 + \cdots \quad \text{where} \\
P_i &= x_i P \quad \text{is the partial pressure and} \\
x_i &= \frac{n_i}{n} \quad \text{is the mole fraction}
\end{aligned}$$

Taking an alternative approach, we can say that the **partial volume**, or **volume percent** as is more commonly used, is the volume V_i that would be occupied by each gas at the same pressure P and temperature T and using the general gas law as above to write:

$$\begin{aligned} V &= \frac{n_1 RT}{P} + \frac{n_2 RT}{P} + \frac{n_3 RT}{P} + \cdots \\ &= \left(\frac{n_1}{n} + \frac{n_2}{n} + \frac{n_3}{n} + \cdots\right)\frac{nRT}{P} \\ &= \left(\frac{n_1}{n} + \frac{n_2}{n} + \frac{n_3}{n} + \cdots\right)V \\ &= V_1 + V_2 + V_3 + \cdots \quad \text{where} \\ V_i &= x_i V \quad \textit{is the percent volume} \end{aligned}$$

As an example, the mole fraction, partial pressure and partial volume of gases in air is listed in Table 5.4. In this comparison, one mole of air has a molecular weight of 28.97 grams and occupies 22.4 liters at 1.00 atmosphere pressure. The mole fraction is the fraction of molecules in the container for each particular gas. The partial pressure and partial volume are then the product of the mole fraction and the total pressure or volume of the mixture.

Elemental Constituent	Mole Fraction	Molecular Mass (kg/kmol)	Molecular Mass in air	Partial Pressure (atm)	Partial Volume (Liters)
Oxygen	0.210	32.00	6.70	0.210	4.693
Nitrogen	0.781	28.02	21.88	0.781	17.492
Carbon Dioxide	0.000	44.01	0.01	0.000	0.007
Hydrogen	0.000	2.02	-	0.000	0.000
Argon	0.009	39.94	0.37	0.009	0.209
Neon	0.000	20.18	-	0.000	0.000
Helium	0.000	4.00	-	0.000	0.000
Krypton	0.000	83.80	-	0.000	0.000
Xenon	0.000	131.29	-	0.000	0.000
Total	1.000		28.97	1.000	22.401

Table 5.4: Partial pressures of gases in air.

Problems

71. An automobile tire holds about 2.0 atm of pressure and has a volume of about 60 liters. How many moles of air will it contain at this pressure and a temperature of 20 degrees C? ans. 0.51 moles

72. A cylinder contains 0.100 moles of Hydrogen at 1.00 atm pressure and 293 degrees K. The piston is slowly pulled outward to increase the volume by a factor of 10 in an isosthermal (constant temperature) expansion. What is the initial volume of the cylinder, the final volume and the final pressure? ans. 2.40liters, 24.0 liters, 0.100 atm.

73. Show that the work done by a gas in an isothermal expansion from V_1 to V_2 is $nRT \ln\left(\frac{V_2}{V_1}\right)$.

74. A gaseous mixture made from 10.0 g of oxygen and 10.0 g of methane is placed in a 10.0 L vessel at 25.0âĹđC. What is the partial pressure of each gas, and what is the total pressure in the vessel? ans. O2 0.765 atm, CH4 1.528 atm, 2.293 atm.

22 Real Gases

The behavior of ideal or perfect gases is given exactly by the ideal gas law, $PV = nRT$; however, the behavior of most real gases especially at high temperatures and pressures can be described only by adding additional terms to the ideal equation. In general terms, most gases are more compressible at low pressures and less compressible at high pressures.

22.1 Compressibility Factor

The best way to describe the deviation of a real gas from the behavior of an ideal gas is with the **compressibility factor**, defined by

$$Z = \frac{PV}{RT} \tag{22.1}$$

for one mole of a gas. The behavior for a real gas is demonstrated in figure 5.2(A) for air and Carbon dioxide. If these gases were ideal, the compressibility factor would be unity at all pressures as indicated by the dotted line. As this graph illustrates, there is significant deviation from ideal behavior in real gases.

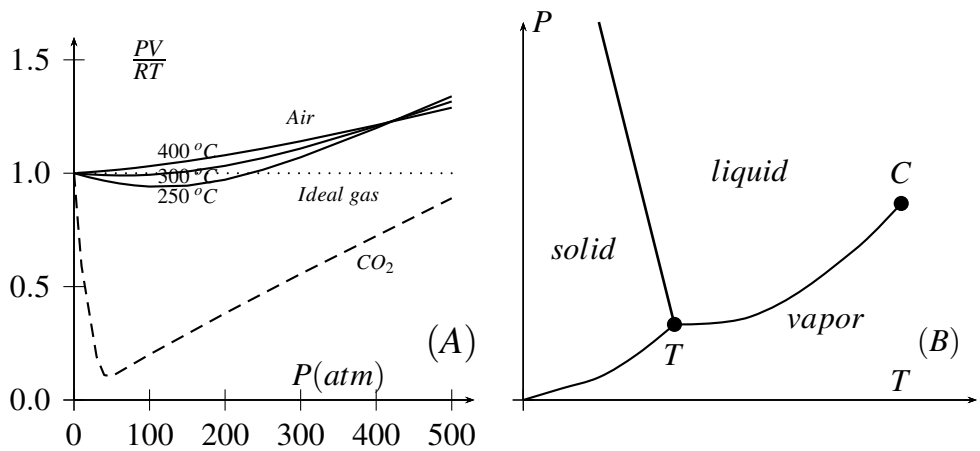


Figure 5.2: (A) Compressibility factor for air and Carbon dioxide and (B) Phase diagram of pure substance.

22.2 Phase Diagrams

A pure substance may exist in the solid, liquid and vapor phases as illustrated in figure 5.2(B) in which the vapor pressure is plotted against the temperature. The solid line separating the solid and liquid regions is known as the **fusion line**, the solid line separating the liquid and vapor regions is known as the **vaporization curve** and the solid line separating the solid and vapor regions is known as the **sublimination curve**. The phases of the substance coexist in equilibrium at all points along their respective lines. The point at which all three phases coexist in equilibrium is labeled with a T and is known as the **triple point**. The point labeled with a C is the **critical point**. Above this point there is no distinction between the liquid and vapor phases. The pressure at this point is known as the **critical pressure** and the volume is known as the **critical volume**. These critical points are listed for several substances in table 5.5. Substances that expand on freezing, such as water, have fusion lines with a negative slope while substances that contract on freezing have a fusion line with a positive slope.

22.3 Saturation Pressure

There is a tendency for molecules of a liquid or solid to evaporate from a surface into the free space above it. Simultaneously, molecules in the vapor above the

substance have a tendency to condense back into the solid or liquid state. When the rate of evaporation equals the rate of condensation, the vapor is in equilibrium with the substance. The vapor is then said to be **saturated** and the pressure exerted by the vapor is known as the **vapor pressure**. This vapor pressure of the substance then adds to the partial pressures of the other gases in the free space above the substance to account for the total pressure.

For example, at sea level with the temperature at 20 ^oC, water vapor has a partial pressures of about 0.023 atm while Nitrogen has a partial pressure of about 0.770 atm, Oxygen has a partial pressure of about 0.207 atm and Argon has a partial pressure of about 0.009 atm. The equilibrium is very dependent upon the temperature with the vapor rising with the temperature. For example, when the vapor pressure reaches 1 atm of pressure, the temperature of the substance is said to have reached the **boiling point** of the substance. The saturated vapor pressure for water is illustrated in figure 5.3. The boiling point at standard temperature and pressure is indicated by dotted lines.

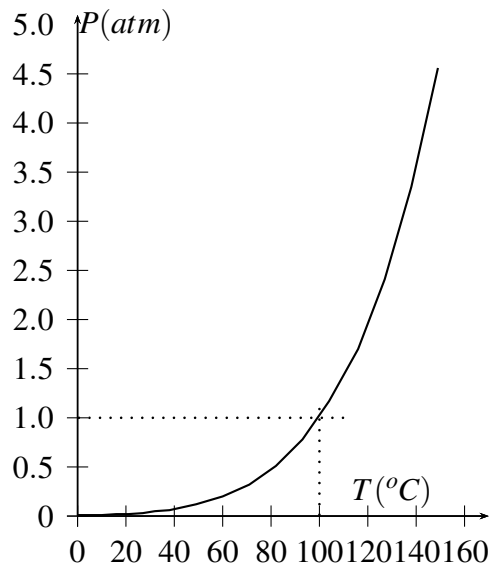


Figure 5.3: Saturated vapor pressure for water.

The saturation pressure, or maximum partial pressure, of water vapor varies with the temperature of the air. Numerous formulas have been used to estimate the saturation pressure. One empirical formula offered by J. A. Goff and S. Gratch in 1946 and revised by Goff in 1957 is considered reliable enough to be used in engineering design calculations over a range from -50 to 102 degrees C.[3]

$$\begin{aligned}\log P = &-7.90298\left(\frac{373.16}{T}-1\right)+5.02808\log\frac{373.16}{T}-1.3816\times10^{-7}\left(10^{11.344(1-T/373.16)}-1\right)\\ &8.1328\times10^{-3}\left(10^{-3.49149(373.16/T-1)}-1\right)+\log(1013.246)\end{aligned} \tag{22.2}$$

[3]Goff, J. A., and S. Gratch "Low-pressure properties of water from -160 to 212 F", Transactions of the American Society of Heating and Ventilating Engineers, pp 95-122, presented at the 52nd annual meeting of the American Society of Heating and Ventilating Engineers, New York, 1946.

Substance	a	b	Critical Temperature	Critical Pressure	Critical Volume	Triple Point	Triple Point
	$(L^2atm/Mole^2)$	$L/Mole$	^oK	atm	liters	^oK	atm
Helium	0.03412	0.02370	5.19	2.241	0.057	2.186	0.0503985
Hydrogen	0.2444	0.02661	33.2	12.80	0.065	13.84	0.06947
Oxygen	1.360	0.03183	154.8	49.78	0.078	54.36	0.0015
Nitrogen	1.390	0.03913	126.2	33.46	0.0895	63.18	0.1237
Water	5.464	0.03049	647.106	217.7	0.056	273.16	0.006117

Table 5.5: Van der Waals constants, Critical and Triple points of common substances.

In this equation the resulting pressure is in hPa, which is equal to 1 mbar, and T is in degrees K. Numerous other equations have been offered since 1946 with varying degrees of success. A term often used is the **humidity** of air. By definition, the relative humidity is the ratio of the partial pressure of water vapor in air to the saturation pressure.

22.4 Real Gas Equations

In 1873, a Dutch physicist, Johannes Diderik van der Waals, was the first to derive an equation to include molecular effects in the equation of state for gases and liquids. He received the Nobel prize in 1910 for this work.

$$\left(P+\frac{a}{V^2}\right)(V-b)=RT \tag{22.3}$$

In this equation, the constant a represents the magnitude of an attraction between two molecules of a gas and the constant b is related to the size of the molecules. When the volume is small the term $\frac{a}{V^2}$ is large and tends to enhance the effect of the pressure making the volume smaller. The constant b is about 4 times the volume of the gas molecule so that when the volume is small the constant b is important and reduces the effective volume. When the volume is large, both the a and b terms are negligible and the Van der Waals equation reduces to the ideal gas equation, $PV = nRT$.

Other equations of state used to describe real gases include the Redlich-Kwong equation, the Berthelot equation, the Dieterici equation and the Clausius equation, all of which are similar to the Van der Waals equation. However, it should be pointed out that although the van der Waals equation is an improvement in the ideal gas equation for certain regions it is not appropriate for quantitative calculations and should be used only qualitatively.

Van der Waals equation can be rewritten in cubic form as

$$V^3 - V^2\left(b + \frac{RT}{P}\right) + V\frac{a}{P} - \frac{ab}{P} = 0 \tag{22.4}$$

Now expanding $(V - V_c)^3$ using the binominal theorem and equating coefficients of like powers of V will allow the Van der Waals constants a and b to be expressed in terms of the critical constants.

$$a = \frac{27R^2T_c^2}{64P_c} \tag{22.5a}$$

$$b = \frac{RT_c}{8P_c} \tag{22.5b}$$

22.5 The Atmosphere

We can think of the atmosphere as being composed of four layers as outlined in table 5.6. There are also thin layers separating each of the main layers a few kilometers thick called the tropopause, the stratopause and the mesopause.

Layer	Altitude km	Temperature ^oC
Troposphere	0-18	17 - -52
Stratosphere	18-60	-52 - -3
Mesosphere	60-120	-3 - -93
Ionosphere	120-600	-93 - +1727

Table 5.6: Layers of the atmosphere

In the troposphere, temperature decreases linearly from 17 to -52 ^oC while pressure decreases exponentially from 1 atm at the earth's surface to about 0.1 atm at the limit of the troposphere. All cloud formation and weather activity takes place in the troposphere. Vertical airflow results from convection of heat upward by airflow.

In the stratosphere, the temperature rises from -52 to -3 ^oC due mostly to absorption of ultraviolet radiation in a thin Ozone layer at the upper limits of the stratosphere. This Ozone layer prevents ultraviolet radiation from reaching the earth's surface where it would damage biological growth. Airflow is mostly horizontal in this layer.

The temperature decreases again in the mesosphere from -3 to -93 ^oC so that the top of the mesosphere is considered the coldest place on earth. The mesosphere lies above the maximum altitude accessible to aircraft and below the minimum altitude of orbital spacecraft. Incoming meteors commence burnup in the mesosphere.

The ionosphere (or thermosphere) is composed largely of ionized gases and is often characterized by its free electron density. It absorbs high energy gamma rays and reflects radio waves back toward the earth. The aurora also takes place in the ionosphere. The temperature rises to about 1800 ^oC and remains constant as altitude further increases, although the extremely rarefied density of the air makes definition of temperature in the conventional sense difficult. The ionosphere contains less than 0.1% of the earth's atmosphere.[4]

22.6 Atmospheric Pressure and Temperature

In the troposhpere the atmospheric pressure and temperature can be modeled using a barometric formula. Consider a plate of area A on which the pressure may be defined as the weight of a column of air above the plate divided by the area. Stated in mathematical terms this becomes

$$dP = -\frac{\rho g A dy}{A} = -g\rho dy \tag{22.6}$$

Using the definition of density $\rho = \frac{mnN_o}{V}$ where m is the average mass of an air molecule, n is the number of moles and N_o is Avagadro's number, we can rewrite equation 22.6 as

$$dP = -\frac{mnN_o g}{V} dy. \tag{22.7}$$

Dividing by P and using equation 21.27 we obtain

$$\frac{dP}{P} = -\frac{mg}{kT} dy, \tag{22.8}$$

which can be integrated to give

$$P = P_o e^{-y/h_o} \quad where \tag{22.9}$$

$$h_o = \frac{kT}{mg} \tag{22.10}$$

[4]Thermal equilibrium between ions exists up to around 400 km making possible a kinetic energy definition of temperature.

In this equation, p_o is atmospheric pressure at sea level (14.696 lb/in^2), m is the mass of one molecule, g is the acceleration of gravity, k is Boltzman's constant, T is the temperature in degrees Kelvin and y is the height above sea level. Variations in the pressure $p(y)$ at a height y above sea level due to small changes in altitude can be estimated from

$$\frac{dP}{P} = -\frac{mg}{kT}dy \tag{22.11}$$

Equation 22.9 works surprisingly well. Without taking account of variations in gravity and temperature with altitude, this equation can be fitted to data obtained in missile flights from White Sands, NM to within less than 3% up to 11 km by taking the value of h_o to be $7,539$ meters. It may be noted that atmospheric pressure drops by a factor of $1/e$ for each $7,539$ meters, ($24,736$ feet) of ascent. A more exact expression can be obtained by noting that the temperature drops off at nearly a linear rate up to about 11 km and can be represented by an equation of the type

$$T = 288 - 0.00650y \tag{22.12}$$

where y is expressed in meters and T in Kelvin. Using this expression for the temperature, equation 22.6 can be integrated to get a somewhat more accurate expression for the pressure

$$P = P_o\left(1 - \frac{By}{T_o}\right)^{\frac{g}{RP_o}} \tag{22.13}$$

Example

Calculate the atmospheric pressure at $6,000$ feet using equation 22.9.

$$P = 14.696\left(e^{-\frac{6000}{24736}}\right) = 11.53\ psi \tag{22.14}$$

This result may be compared to the measured value of 11.78 psi.

Problems

75. Calculate the van der Waals constants a and b and compare to those reported in tables. ans. $a = 1.3667$ and $b = 0.03189$

76. Calculate the atmospheric pressure at 18 km. ans. 0.091 atm

77. To what height will a column of Mercury rise in an evacuated tube due to the pressure of the atmosphere at sea level? ans. 763 mm

78. To what height will a column of Water rise in an evacuated tube due to the pressure of the atmosphere at sea level? ans. 33.9 ft

79. What is the average kinetic energy of an ion at a temperature of 1800 ^oC? ans. 4.29×10^{-20} Joules or 0.268 eV.

80. Estimate the saturation pressure of water at 20, 40 and 60 degrees C using both the Goff and Kratch and the Goff formulas.

23 Liquids

We distinguished between liquids and gases by noting that liquids are generally incompressible while the volume of gases are sensitive to even small changes in pressure and temperature. In keeping with that distinction, this section is devoted to generally incompressible fluids.

23.1 Water Pressure

The pressure below sea level is equal to the pressure of the column of water above the selected depth added to the pressure of the atmosphere at sea level.

$$p(h) = p_o + \rho g h \tag{23.1}$$

At equilibrium, the pressure at any point in a fluid is the same in all directions and the pressure is the same at all points of the same elevation or depth.

Example

Calculate the pressure on a spherical diving bell $5,000$ feet below the surface of the sea, the stress developed in the walls of the diving bell and the decrease in the diameter. Assume the diving bell has an inside radius of 5 feet and wall thickness

of 0.5 ft. Is the diving bell near crush depth?

$$p_{5000} = 14.696\ psi + \frac{62.43\ psf)(5000\ ft}{144} = 2167 psi \tag{23.2}$$

$$\sigma = \frac{pr}{2h} = \frac{(2167)(60)}{(2)(6)} = 1.08 \times 10^4\ psi \tag{23.3}$$

$$\varepsilon = \frac{\sigma}{Y} = \frac{10,833}{30.6 \times 10^6} = 0.0354\% \tag{23.4}$$

The fractional decrease in the circumference of 0.0354% will result in a decrease in the radius by $\frac{C-0.000354C}{2\pi} = 0.052\ in$. The yield strength of steel is 90 MPa = 1.3×10^4 psi, which is about 30% greater than the stress being developed in the hull.

23.2 Pascal's Principle

Under equilibrium conditions, a change in pressure (or a pressure applied to an enclosed fluid) is transmitted uniformly and equally throughout the fluid to all points of the fluid and the walls of the containment vessel. This principle is attributed to Blaise Pascal (1622-1663), a French mathematician, physicist, inventor, writer and Catholic philosopher.

If a force F_d is applied to the small piston in figure 5.4 with area a, the pressure developed in the fluid will be $p = \frac{F_d}{a}$. This pressure will be transmitted throughout the fluid and will exert a pressure on the large piston with area A. The large piston will then exert a force $F_u = pA = \frac{A}{a}F_d$ on the weight $W = mg$. If the net force $(pA - W)$ is greater than zero, the weight will be pushed upward. The fluid is incompressible; therefore, the rate at which the weight is moved will depend on the rate at which the small piston is pushed inward. If the small piston is pushed inward with a speed of V_a, the large piston will move upward at a rate $\frac{a}{A}V_a$. This is the exact principle by which a hydraulic lift in an automobile service station works.

$$F_u = \frac{A}{a}F_d \tag{23.5}$$

$$d_u = \frac{a}{A}d_d \tag{23.6}$$

Normally, the small piston is replaced by a hydraulic pump since moving the large piston any distance would require movement of the small piston larger distances than can be easily allowed in most mechanical systems.

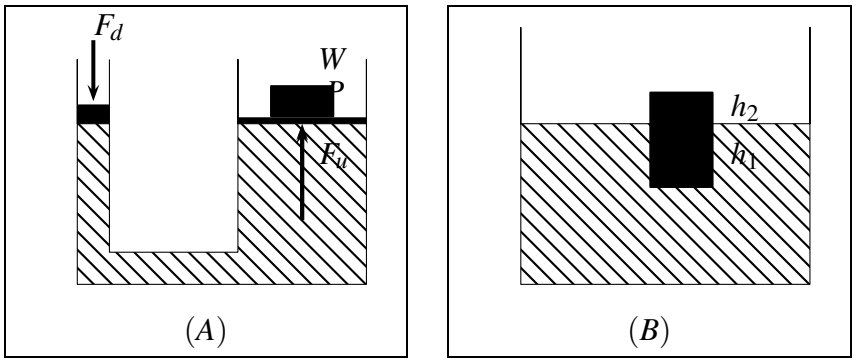


Figure 5.4: Hydraulic lift (A) and Archimedes principle (B).

Example

What pressure is developed in a hydraulic system to lift a $3,000$ automobile when the diameter of the driving cylinder is 2.0 in and that of the lifting cylinder is 1.00 foot? The upward force necessary to lift the car is $3,000$ lb and the area of the piston is $\pi d^2/4 = 0.785$ so the pressure needed to provide that lift is

$$p = \frac{3,000\,lb}{0.785\,ft^2} = 3.82 \times 10^3\,\frac{lb}{ft^2} \tag{23.7}$$

23.3 Archimedes' Principle

A body of surface area A and density ρ_b wholly or partly submerged is a fluid of density ρ_w is buoyed upward by a force equal to the weight of the fluid displaced by the body. This principle may be used to determine the specific gravity of the body. This principle is known as **Archimedes principle** was discovered by Archimedes of Syracuse, a Greek mathematician, physicist, engineer, inventor, and astronomer (287 BC-212 BC).

If the body is submerged a distance h_1, the buoyant force will be $\rho_w h_1 A$. The weight of the body acting downward will be $\rho_b A(h_1 + h_2)$. At equilibrium, these forces will balance allowing the density to be calculated.

$$\rho_b A(h_1 + h_2) = \rho_w h_1 A \tag{23.8}$$

$$\rho_b = \rho_w \frac{h_1}{h_1 + h_2} \tag{23.9}$$

If the fluid is water, the specific gravity is the ratio of the densities.

Example

A squared pine cant floats in a pond with 90.0% of its mass under water. If the density of water is taken as 62.4 lb/ft^3, the density of pine from equation 23.9 will be

$$\rho_{pine} = (0.90)(62.4\,\frac{lbs}{ft^3}) = 56.2\,\frac{lbs}{ft^3} \tag{23.10}$$

23.4 Stoke's Law

Sir George Gabriel Stokes (1819-1903), an Irish mathematician and physicist, obtained a solution to the Navier-Stokes equation for the viscous drag on a spherical body in free fall through a viscous liquid as illustrated in figure 5.5.

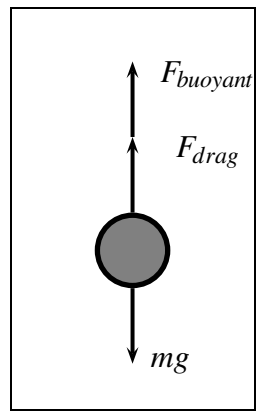


Figure 5.5: Sedimentation in a viscous fluid.

$$F_d = 6\pi r\eta V \tag{23.11}$$

In this equation, known as **Stoke's law**, r is the radius of the sphere, V is the velocity and η is the viscosity of the fluid. This law is particularly useful in obtaining the sedimentation velocity of an object falling through air at low velocities. The sedimentation velocity is reached by an object falling through air when the upward force due to the viscous drag $6\pi r\eta V_s$ and the buoyant force of the air displaced by the object, F_d, balances the downward force of gravity, mg. If, for example, the object is spherical $F_d = \frac{4}{3}\pi r^3\rho_{air}g$ and $mg = \frac{4}{3}\pi r^3\rho_{body}g$, the sedimentation velocity can be calculated from

$$\frac{4}{3}\pi r^3\rho_{body}g = \frac{4}{3}\pi r^3\rho_{air}g + 6\pi r\eta V_s \tag{23.12}$$

$$V_s = \left(\frac{2r^2}{9}\right)\frac{\rho_{body}-\rho_{air}}{\eta}g. \tag{23.13}$$

Stokes' law may be used to predict the settling velocity of small spheres in liquids and gases moving at low velocities when the drag force due friction can be ignored. In measurements of the sedimentation velocity, it is often convenient to use a centrifuge in which the gravitational force is replaced by the centrifugal force $mr^2\omega$.

Problems

81. Scuba divers use a pressure regulator that allows air from a tank to enter the lungs at the pressure of the sea. Nitrogen narcosis sets in at a depth of about 10.0 meters. What is the sea pressure at this depth? ans. 28.9 psi or 200 kPa

82. What is the water pressure at the bottom of the Mariana trench 11 km (36,091 ft) below the surface of the ocean. ans. 15,654 psi or 108,011 kPa

83. What would the stress be in a spherical diving bell of radius 2 ft and wall thickness of 2 inches at this depth be? ans. 93,924 psi

84. Hydraulic fluid is pumped into a hydraulic pump through an opening of 1/4 inch diameter at a pressure of 50.0 psi. If the diameter of the pump cylinder is 3 inches, what is the pressure developed in the hydraulic piston. ans. 7209 psi

85. Ice has a density of 0.9168 kg/m^3. How much of an iceberg will be exposed above water if the density of sea water is 1.025 kg/m^3? ans. 11.56%

86. What is the sedimentation velocity of a small, spherical particle of iron ore with density 5000 kg/m^3 and diameter of 2 micrometers in water where the density is 1000 kg/m^3 and viscosity is 0.001 Pa sec? ans. 8.72 micrometers/sec

87. What is the density of wood which floats in fresh water with $2/3$ of its volume submerged? What is the specific gravity of the block? What would be the buoyant force of water on the block of wood if it has a volume of 10.0 ft^3 and is completely submerged? ans. 41.6 lb/ft^3, 0.667, 624 lbs.

24 Bernoulli's Equation

Bernoulli's principle, named after the Dutch-Swiss mathematician Daniel Bernoulli who published his principle in 1738, states that an increase in the flow rate is accompanied by a decrease in the pressure. This principle can be expressed in terms of an equation by considering the flow rate and pressure of a fluid in a pipe with a constriction as illustrated in figure 5.6.

A volume element of fluid $A_1\Delta x_1$ traveling at a velocity V_1 flows from the lower part of the pipe to the upper part where it is contained in an element $A_2\Delta x_2$ traveling at a velocity V_2. The volume of each element is the same, $A_1\Delta x_1 = A_2\Delta x_2$. Since the fluid is incompressible, the mass of the fluid in both elements is the same $M = \rho A_1\Delta x_1 = \rho A_2\Delta x_2$. This statement, after canceling the densities, dividing by Δt and using $V = \frac{\Delta x}{\Delta t}$, is often referred to as the **equation of continuity** for incompressible fluids.

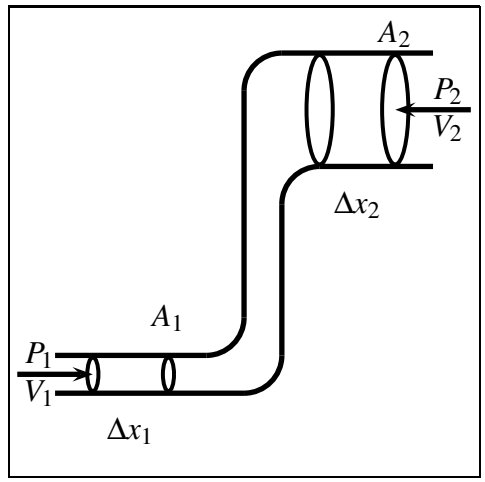


Figure 5.6: Fluid flow through pipe with changing diameter and height.

$$V_1A_1 = V_2A_2 \tag{24.1}$$

The driving force exerted by fluid pressure at the lower end is P_1A_1 while the opposing force due to pressure in the fluid at the upper level is P_2A_2; so that the work done in moving the volume element is

$$W = P_1A_1\Delta x_1 - P_2A_2\Delta x_2 \tag{24.2}$$

The change in kinetic and potential energies of the mass of the fluid can be written as

$$\Delta K = \frac{1}{2}M\left(V_2^2 - V_1^2\right) \tag{24.3}$$

$$\Delta U = Mg\left(y_2 - y_1\right), \tag{24.4}$$

where y_1 is the level of the lower part of the pipe and y_2 is the level of the upper part. Since the change in total energy $E = K + U$ equals the work done in moving the fluid through the pipe

$$P_1A_1\Delta x_1 - P_2A_2\Delta x_2 = \frac{1}{2}M\left(V_2^2\right) - \frac{1}{2}M\left(V_1^2\right) + Mgy_2 - Mgy_1 \tag{24.5}$$

Collecting like terms on either side and replacing the volume elements $A\Delta x$ by M/ρ of the equation gives

$$P_1\frac{M}{\rho} + \frac{1}{2}M\left(V_1^2\right) + Mgy_1 = P_2\frac{M}{\rho} + \frac{1}{2}M\left(V_2^2\right) + Mgy_2 \tag{24.6}$$

Multiplying through with ρ, canceling the mass from each equation and noting that for this equation to hold true each side must equal a constant, we can drop the subscripts and write the result in the form known as **Bernouli's equation** for steady, non-viscous, incompressible flow.

$$P + \frac{1}{2}\rho V^2 + \rho g y = constant \tag{24.7}$$

A number of phenomena experienced in daily life can be explained by Bernoulli's equation when used with the equation of continuity.

24.1 Nozzle Velocity

If the output is at the same level as the input, the output opening is made much smaller than the input and the fluid is allowed to exit the pipe to the atmosphere, Bernoulli's equation becomes

$$P_{line} - P_{atmos} = \frac{1}{2}\rho\left(V^2 - V_1^2\right) \tag{24.8}$$

so that in the limit of $P_{line} >> P_{atmos}$ and $A_2 << A_1$, the exit velocity V_2 can be written as

$$V_2 \longrightarrow \frac{A_1}{A_2}\sqrt{2\rho P} \tag{24.9}$$

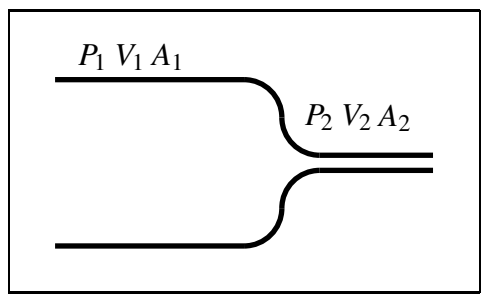


Figure 5.7: Fluid flow through Nozzle.

This shows that the exit velocity of water, for example, from a restricted pipe opening is proportional to the square root of the pressure multiplied by the ratio of the pipe area to the nozzle area, an effect often observed in garden hoses. The smaller the opening, or tighter the nozzle, the higher the velocity of the water.

24.2 Water Tank

If the water is held in a tank where the upper end is open to the atmosphere and a valve at the bottom is also open to the atmosphere, Bernoulli's equation becomes

$$P_o + \frac{1}{2}\rho V_1^2 + \rho g y_1 = P_o + \frac{1}{2}\rho V_2^2 + \rho g y_2 \tag{24.10}$$

Substituting $h = y_1 - y_2$ and noting that $V_2 = \frac{A_1}{A_2}V_1 >> V_1$, this result can be rewritten as

$$V_2^2 = 2gh, \tag{24.11}$$

which explains why water falls further from the tank when the tank is full and why a water tank must have a nozzle to direct the flow of water to its intended target.

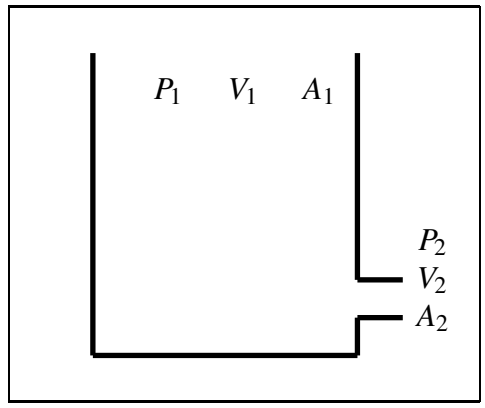


Figure 5.8: Water flowing from tank.

24.3 Airplane Wing Lift

Bernoulli's equation written for air passing over an airplane wing shows why the wing produces lift.

$$P_1 + \frac{1}{2}\rho V_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho V_2^2 + \rho g y_2 \tag{24.12}$$

Since the distance from front to back over the top of the wing is greater than the distance under the wing, the velocity over the top V_2 must be greater than the velocity under the wing V_1 so we can write $V_2 = kV_1$. Noting that the two potential energy terms are essentially equivalent allows us to write

$$P_1 - P_2 \simeq \frac{1}{2}\rho V_1^2 \left(k^2 - 1\right) \tag{24.13}$$

As an example, assume k=2, ρ = 0.002378 $slugs/ft^3$, that the wings have a total surface area of 80 sq. ft. and that the airplane is flying at a speed of 150 mph or 220 ft/sec. In this case

$$\Delta P = \frac{1}{2}(0.002378)(220)^2(3) = 172 psf \tag{24.14}$$

$$Lift = 172(80) = 13,811 lb. \tag{24.15}$$

Also, a typical wing will have a thickness of about 10 $in = 0.83\ ft$. Assuming an airspeed of 150 mph (220 ft/sec) the pressure difference between the top and bottom of the wing calculated from equation 24.13 can be approximately 150 lb/ft^2. With a total wing area of 80 ft^2 approximately 12,000 lb can be generated. Obviously, calculating the aerodynamic lift of a wing involves other considerations, but this is a clear demonstration of how a wing can generate enough lift to maintain flight.

24.4 Pump Head

The term "**pump head**" is widely used by pump makers and widely misunderstood by pump users. In simple terms, pump head pressure P_H is that pressure which will raise water to a level y above the pump. The maximum height to which the pump can raise water but not maintain flow is normally called the "shut-off head". Sometimes pump head is expressed in pounds per square inch (or foot) and sometimes in terms of the work per unit length the pump can do $\frac{P}{\rho g}$. From a mathematical standpoint, pump head can be explained by setting $P_2 = 0$ and $V_1 = V_2 = 0$ in Bernoulli's equation to get

$$P_1 + \rho g y + 0 = 0 + \rho g y_2 + 0 \quad \text{so that} \tag{24.16}$$

$$P_1 = \rho g (y_2 - y_1) \tag{24.17}$$

Inserting values of density and gravity in different units gives the relation between output pressure and pump head in different systems.

$$P[Pa] = 9810 h[m] \tag{24.18}$$

$$P[psf] = 62.4 h[ft] \tag{24.19}$$

$$P[psi] = 0.434 h[ft] \tag{24.20}$$

24.5 Venturi Meter

The pressure of a fluid is reduced when a fluid flows through a constricted section of pipe. This reduction in pressure is known as the Venturi effect, named after an Italian physicist Giovanni Battista Venturi (1746-1822). The Venturi effect can be used to measure volumetric flow rates of incompressible fluids. Designating the cross-sectional area of the input pipe by A and that of the constriction as a and writing Bernoulli's equation for the flow at the input and constriction gives

$$P_1 + \frac{1}{2}\rho V_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho V_2^2 + \rho g y_2. \tag{24.21}$$

In this equation, V_1 refers to the velocity of the fluid at the input and V_2 refers to the velocity in the constriction. From the equation of continuity we see that $V_2 = \frac{A}{a} V_1$, where A and a refer to the

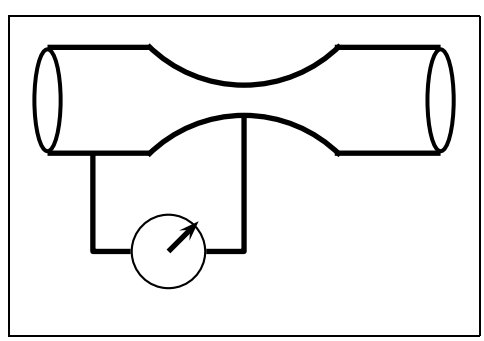


Figure 5.9: Venturi meter.

area of the input and the constriction respectively. Noting that $y_1 = y_2$ we see that this equation can be rewritten as

$$V_1 = \sqrt{\frac{2\Delta P}{\rho(b^2-1)}}, \qquad (24.22)$$

where $b = A/a$ and $\Delta P = P_1 - P_2$.

24.6 Pitot Tube

The **Pitot tube** was invented by the French engineer Henri Pitot in the early 1700*s* and was modified to its modern form in the mid 1800*s* by French scientist Henry Darcy. It can be applied to determine the airspeed of an aircraft and to measure air and gas velocities in industrial applications. A Pitot tube, illustrated schematically in figure 5.9, consists of a static air source B at the same temperature as the ram air input A. The dynamic pressure of the ram air forces the Mercury level down in the ram air tube. This difference may be measured and related to the airspeed. Considering that static air monitor is at the same elevation as the Pitot tube and that the velocity of the static air is zero, Bernoulli's equation may be applied to both the dynamic and static air to get

$$P_d + \frac{1}{2}\rho V_d^2 + \rho g y_d = P_s + \frac{1}{2}\rho V_s^2 + \rho g y_s, \qquad (24.23)$$

where the subscripts s and d represent static and dynamic variables and ρ the density of air. The difference between the dynamic and static pressure is measured by the difference in the height of the Mercury column h in the two tubes.

$$P_d = P_s + \rho' g h \qquad (24.24)$$

where, ρ' represents the density of the Mercury. Combining these two equations and solving for the dynamic velocity gives

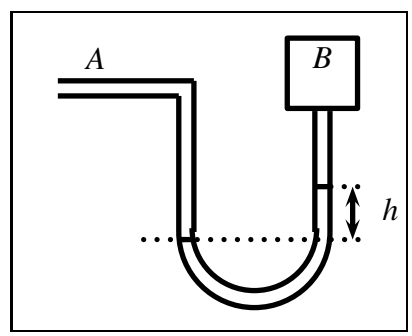


Figure 5.10: Pitot tube.

$$V_d = \sqrt{\frac{2\rho' g h}{\rho}}. \qquad (24.25)$$

Problems

88. What is the velocity of water ejected from a nozzle with a diameter of 0.200 inches connected to a hose of diameter 1.00 inch when the water pressure in the hose is 60.0 psi? ans. 51.1 ft/sec

89. What is the velocity of water spurting from a bullet hole in a water tank 3.00 meters from the top of the water? ans. 7.67 m/sec

90. What is the lift of the wings of an airplane flying at 175 mph and having a total wing surface of 50.0 sq. ft.? Assume the distance across the top of the wing is twice the distance under the wing. ans. $1.175\ x\ 10^4$ lb

91. What is the pump head of a pump that will deliver 100 psi pressure? ans. 230 ft.

92. What is the speed of air flowing through a venturi tube if the ratio of the area of the input to the constriction is 4.00 and the pressure differential developed is 5.00 psi? ans. 201 fps

93. What is the airspeed indicated by a pitot tube if a difference of 1.00 cm in the levels of Mercury is observed in a u-tube? ans. 46.55 m/sec

25 Molecular Forces in Liquids

At the molecular and atomic level, certain short range forces tend to bind molecules together. For molecules of the same kind the force of attraction is called **cohesive**; for molecules of different types the force of attraction is called **adhesive**. For example, rain falls in droplets instead of in a mist because the cohesive forces between water molecules bind them together. In another case, water poured on clean glass tends to spread, forming a thin, uniform film over the glass surface "wetting" the glass because the adhesive forces between water and glass are stronger than the cohesive forces between the water molecules. In contrast to water, Mercury has stronger cohesive forces and will not "wet" a glass surface. Instead it will "roll" along a glass surface.

One way to illustrate the effects of the forces of cohesion and adhesion is to suspend a small capillary tube in a beaker filled with water and another in a beaker filled with Mercury, as shown in figure 5.11. In the first illustration, water rises in a capillary tube made from paraffin with the liquid making contact

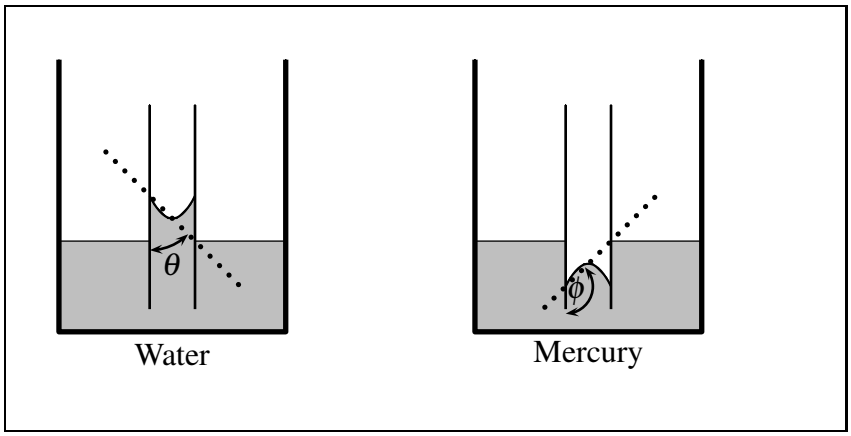


Figure 5.11: Surface tension forces acting in a capillary tube.

with the surface at an angle $\theta = 107^o$ because the adhesive force between water and paraffin is stronger than the cohesive force between water molecules. In the second illustration, the level of the Mercury column is depressed with the liquid Mercury making contact with the surface of a capillary tube made from glass at an angle $\phi = -139^o$ because the cohesive force between atoms of Mercury is stronger than the adhesive forces between Mercury and glass. If the angle is positive the surface of the meniscus will be concave upward; but if the angle is negative the surface of the meniscus will be concave downward.

25.1 Surface Tension

This phenomena can be described by defining **surface tension** acting at right angles to the line between the liquid and the glass and equal to the ratio of the molecular force acting to the length of the line. Surface tension, represented by γ is normally expressed in cgs units because of its small size.[5]

Water will rise in the capillary tube until the upward pull of surface tension is balanced by the pull of gravity downward on the column of water. The force of gravity acting downward is $W = \pi r^2 \rho g h$ where ρ is the density of the water, g is the acceleration of gravity and h is the height to which the column of water rises. The force upward due to surface tension is $2\pi r \gamma \cos\theta$ where $2\pi r$ is the length of the line at the points where the water contacts the glass surface, γ is the surface tension and θ is the angle of contact. The surface tension and the height to which

[5] 1 dyne = 10^{-5} N = 2.248×10^{-6} lb

the column of water rises can be calculated as follows:

$$\pi r^2 \rho g h = 2\pi r \gamma \cos\theta \tag{25.1}$$

$$\gamma = \frac{\rho g r h}{2\cos\theta} \tag{25.2}$$

$$h = \frac{2\gamma\cos\theta}{r g \rho} \tag{25.3}$$

The surface tension of some common liquids is listed in table 5.7

Liquid	Temp ^oC	Contact Angle w Glass degrees	Surface Tension dynes/cm	Viscosity mPa
Ethyl Alcohol	20	0	22.3	1.200
Water	20	0	72.8	1.002
Glycerin	20	0	63.1	1410
Mercury	25	139	473	
Blood	37		55.89	4
Oil(SAE 10)	30			200
Soap solution		26		

Table 5.7: Surface tension of some common substances.

One of the more interesting applications of surface tension lies in the understanding of the size of rainwater droplets. In the formation of water droplets, the surface tension tends to reduce the surface area while the difference in pressure inside and outside the droplet resists surface area shrinkage. The shrinking force of surface tension around the equatorial line of a drop is $2\pi r\gamma$, where r is the radius of the droplet. The resisting force acting parallel to the polar axis may be taken from equation 18.7.

$$2\pi r\gamma = \pi r^2 \Delta p \tag{25.4}$$

Equating these two expressions gives an expression relating the pressure difference to the radius of the droplet and the surface tension of the liquid.

$$\Delta p = \frac{2\gamma}{r} \tag{25.5}$$

This equation applies for a droplet comprised entirely of liquid. If the droplet has two surfaces, such as for example a soap bubble, the surface tension will act over both perimeters and result in the expression

$$\Delta p = \frac{4\gamma}{r} \tag{25.6}$$

Surface tension can explain many things observed in nature. A few of these include the ability of some insects to "walk" on water, congealing of water in a stream from a water fountain into droplets, the "pooling" or "puddling" of water on a surface, the transpiration of water through the trunks and leaves of plants, the formation of oil droplets in water, etc. The same concepts are also applicable in the case of Mercury and other liquids.

Examples

Suppose that the inside diameter of the capillary tube suspended in water is 2 mm. Then the height to which water will rise in the capillary tube is obtained making calculations in the cgs system with θ = 0 and γ=73 dynes/cm is found to be

$$h = \frac{(2)(73)(\cos 0)}{(0.1)(980)(1.0)} = 1.5\ cm. \tag{25.7}$$

and in the case of Mercury with θ = 139 and γ=473 dynes/cm

$$h = \frac{(2)(473)(\cos 139)}{(0.1)(980)(13.6)} = -0.54\ cm \tag{25.8}$$

Raindrops normally have a maximum diameter of about 4 mm. Given the surface tension of 73 dynes/cm, calculate the pressure difference between the inside and outside of the raindrop.

$$\Delta p = \frac{(2)(73)}{(0.2)} = 730\ dynes/cm^2 = 0.000721 atm \tag{25.9}$$

Soap bubbles are considerably larger but have a lesser surface tension due to reduction by the detergent in the soap. In the case of a soap bubble of the diameter 2 cm and surface tension of about 30 dynes/cm, the pressure difference would be

$$\Delta p = \frac{(4)(30)}{1} = 120\ dynes/cm^2 = 0.000118 atm \tag{25.10}$$

25.2 Viscosity

Viscosity is a measure of the resistance of one layer of a liquid flowing over another. There are, however, several types of viscosity that are used in different ways and are defined differently. This often leads to confusion so reports of the "viscosity" of a fluid must be examined carefully. These definitions are listed below and always refer to laminar and not turbulent flow.

1. **Dynamic (or Absolute) Viscosity** is the tangential force per unit area required to move one horizontal plane with respect to the other at unit velocity while maintained at unit distance apart. Dynamic viscosity is commonly used by physicists, chemists, mechanical and chemical engineers and is often identified by the symbol μ. The units of viscosity are force $\times$ distance divided by Area $\times$ speed. In cgs units this unit is called the **Poise**, named after the French scientist Jean Louis Marie Poiseuille in $1797-1869$. It is sometimes expressed in terms of centipoise = 0.01 poise.[6] The coefficient of viscosity μ can be measured from the time t required for a given volume V of the liquid to flow through a capillary tube of length L and radius r under an applied pressure P using the formula

$$\mu = \frac{P\pi r^4 t}{8VL} \tag{25.11}$$

The total flow rate through a pipe of length L and radius R is inversely proportional to the viscosity but directly proportional to the pressure gradient $\frac{\delta p}{L}$ and the fourth power of the radius. Poiseuille derived the following equation for calculation of the flow rate.

$$\frac{dV}{dt} = \frac{\pi}{8}\frac{R^4}{\mu}\frac{\Delta p}{L} \tag{25.12}$$

2. **Kinetic Viscosity** is defined as the dynamic viscosity divided by the density of the liquid. The cgs unit for kinematic viscosity is the stoke (St), named after George Gabriel Stokes (1819-1903). It is often represented by the symbol ν and sometimes expressed in terms of centistokes (cSt).[7]

$$\nu = \frac{\mu}{\rho} \tag{25.13}$$

3. **Shear viscosity** describes the reaction of the fluid to applied shear stress. It is the ratio between the horizontal pressure exerted on the surface of a fluid to the change in velocity of the fluid with depth in the fluid (velocity gradient).

[6] $1\ Poise = 100\ cP = 1\ dyn-sec/cm^2 = 1\ g/(cm-sec) = 0.1\ Pa-sec = 0.02089\ slug/(ft-sec)$.

[7] $1\ St = 100\ cSt = 1\ cm^2/sec = 0.0001\ m^2/sec = 0.001076\ ft^2/sec$.

4. **Volume viscosity** (also called bulk viscosity) is important only in case of fluid compression such as in shock waves and sound propagation.

The following examples will provide illustrations of the usefulness of the concept of viscosity and **Poisseuille's equation** in everyday living. It is unfortunate that the units of viscosity lead to considerable difficulty in making calculations.

Example

Calculate the flow rate of SAE 20 motor oil of viscosity 125 cP through a camshaft having an inside radius of 1 cm and length of 10 cm under a pressure differential of 10 lb/in^2.

$$\frac{dV}{dt} = \frac{\pi}{8}\frac{(1)^4}{1.25}\frac{689500}{10} = 21,650\ cm^3/sec \tag{25.14}$$

Poisseuille's equation also tells us that increasing the radius of an artery by a factor of 2 would increase the blood flow rate for the same heart pressure by a factor of 16. The reason for angioplasty or bypass surgery is to obtain an increase in the diameter of the arteries. For example, the flow rate of blood with a dynamic viscosity of 3.5 cP through an artery of 1 mm diameter and 10 cm length under a pressure of 100 mm above normal atmospheric pressure (or about 1.9 psi) would have a flow rate of

$$\frac{dV}{dt} = \frac{\pi}{8}\frac{(0.05)^4}{0.035}\frac{133000}{10} = 0.932\ cm^3/sec \tag{25.15}$$

Increasing the diameter to 2 mm would increase the flow rate to 14.9 cm^3/sec.

Problems

94. The surface tension of blood can be taken as 55.9 dynes/cm, the density as 1.05 gm/cm^3 and the angle of contact as zero with a capillary blood vessel. To what height will blood rise in a capillary blood vessel of radius 2 micrometers? ans. 543 cm

95. What is the difference in pressure between the inside and outside of a soap bubble having a radius of 2.00 cm? Take the surface tension of a soap solution to be 0.029 N/M. ans. 5.8 x 10^{-5} atm

96. Calculate the excess pressure inside a drop of Mercury having a diameter of 3 mm and a temperature of 25 degrees C. ans. 6307 dynes/cm

97. What force is needed to overcome the force of surface tension in removing a wire ring of diameter 3.00 cm from water? 0.01375 N

98. What is the volume flow rate for oil of viscosity 125 cP through a tube 100 cm long with a radius of 1.00 cm when the pressure differential is 10.0 psi? ans 2165 cm^3/sec

Chapter 6

HEAT AND THERMODYNAMICS

A system comprised of many molecules may be described in terms of **microscopic variables** such as the position, velocity and acceleration of each of its molecules; or it may be described in terms of **macroscopic variables** such as pressure, volume and temperature. Thermodynamics is the study of the system in terms of macroscopic variables as they relate to the internal state of the system. The purpose of our study of thermodynamics is to find relations between these thermodynamic variables that are consistent with the fundamental laws of physics. In order to understand thermodynamics, we must first establish conceptual understandings of systems that will be in marked contrast to the microscopic approach undertaken in the study of mechanics.

An understanding of the concept of equilibrium is essential to the study of thermodynamics as it will be undertaken in terms of thermodynamic coordinates. There are three types of equilibrium in a system: mechanical, chemical and thermal. When conditions for all three types of equilibrium are satisfied, a system is said to be in **thermodynamic equilibrium**.

1. When there is no unbalanced force in the interior of a system and between the system and its surroundings, the system is said to be in a state of **mechanical equilibrium**.

2. When a system in mechanical equilibrium does not tend to undergo spontaneous changes of internal structure such as chemical reactions, diffusion or material movement from one part of the system to another, the system is said to be in **chemical equilibrium**.

3. When a system in mechanical and chemical equilibrium undergoes no spon-

taneous change in its thermodynamic coordinates when separated from its surroundings, it is said to be in **thermal equilibrium**.

When two or more systems are in thermal equilibrium with one another, these systems possess a property that ensures their being in thermal equilibrium with one another. This property is called temperature. **Temperature** can be thought of as the property of a system that determines whether or not a system is in thermal equilibrium with other systems. There are other thermodynamic variables, such as pressure and volume, and these coordinates may differ in systems that are in thermal equilibrium; but the temperature of two systems in thermal equilibrium will always be the same.

A **thermodynamic process** is a an continuous evolution from one state to another. There are two types of processes.

1. A **reversible process** is one in which the system is always in equilibrium with its surroundings. Normally processes that are small and made slowly are reversible.

2. An **irreversible process** is one in which the system is not always in equilibrium with its surroundings. Normally an irreversible process occurs when the change in thermodynamic coordinates is large or sudden.

A system may be taken from an initial state to a final state by several processes. Four of the most common processes are defined in table 6.1.

Process	Defining Criteria	Condition
Isobaric	Constant pressure	$dP = 0$
Isochoric	Constant volume	$dV = 0$
Isothermal	Constant Temperature	$dT = 0$
Adiabatic	Without Heat Transfer	$dQ = 0$

Table 6.1: Thermodynamic processes.

Experiment shows that any system in thermodynamic equilibrium may be described in terms of three thermodynamic coordinates; pressure, volume and temperature (P,V,T) The relation between these variables may be written implicitly as $f(P,V,T) = 0$ or explicitly by defining one of the coordinates in terms of the other such as for example $V = f(P,T)$. In this example, we can differentiate the function to obtain

$$dV = \left(\frac{\partial V}{\partial T}\right) dT + \left(\frac{\partial V}{\partial P}\right) dP. \tag{25.1}$$

Then if we define the isobaric volume expansivity by

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \tag{25.2}$$

and the isothermal volume compressibility by

$$k = \frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T, \tag{25.3}$$

the differential equation for dV can be written as

$$dP = \beta B dT - \frac{B}{V} dV. \tag{25.4}$$

This equation will have use in thermodynamics in several ways. For example suppose a mass of Mercury is held at a pressure of $P_i = 1\,atm$ and a temperature of 0^oC and kept at constant volume while the temperature is raised by 10 degrees. Using the volume expansivity coefficient of $1.81 \times 10^{-4}\,deg^{-1}$ and a volume compressibility coefficient of $2.50 \times 10^{11}\,dynes/cm^2$, we can calculate the final pressure in the system to be $P_f = 450\,atm$.

The fundamental relationship between the thermodynamic variables which describe a system is called an **equation of state**. All the thermodynamic variables in an equation of state may be independent except for one. We have already identified the equation of state for an ideal gas, $PV = nRT$ which embodies three other laws of physics, Boyle's law (PV/T = constant), Charles' law ($V = kT$ at constant pressure) and Gay-Lussac's law ($P = kT$ at constant volume). However, real gases only obey this law approximately. Several other equations of state more accurately describe real gases and include those of Van der Waals (1873), Beattie-Bridgeman (1928) and Redlich & Dworg (1949). We can visualize a system as being represented by a point in a three-dimensional coordinate system where the coordinates are pressure P, volume V and temperature T as illustrated in figure 6.1. The points labeled $(P_1V_1T_1)$ and $(P_2V_2T_2)$ represent the system at two possible points on the diagram.

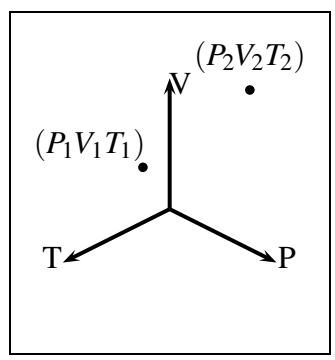


Figure 6.1: Thermodynamic system on PVT diagram.

We can also distinguish between thermodynamic variables by defining a **state function** is a thermodynamic variable which depends only on the current state of the system, not the path taken to reach that state, and a **process function** which does depend on the path taken between two states.

26 Internal Energy

We will define the **internal energy** of a system as the total kinetic energy of the molecules of the gas. Defined this way, **the internal energy is a state variable dependent only upon the temperature of the system and not the path by which the system was brought to that state**. In case the system is monoatomic and the molecules have only translational kinetic energy the total internal energy may be defined using the definitions in section 21.1 and the ideal gas law as follows:

$$U = \frac{1}{2} n N_o m \overline{V^2} = \frac{3}{2} nRT \tag{26.1}$$

In this formula, n is the number moles of the gas, N_o is Avagadro's number, m is the mass of each molecule, ρ is the density of the system and $\overline{V^2}$ is the root mean square velocity of the molecules.

Problems

99. One mole of air occupies a volume of 10.00 liters at a temperature of 400.00 degrees K. Air is composed by volume of Nitrogen (78.09%), Oxygen (20.95%), Argon (0.933%) and Carbon dioxide (0.0300%) with molecular weights of 28.02, 32.00, 39.94 and 44.01 respectively. Calculate the total internal energy of the gas and the average kinetic energy of its molecules. ans. 4988.68 J, 8.28×10^{-21} J

100. Calculate the total internal energy of the Nitrogen molecules. ans. 3895.66 J

101. Calculate the total internal energy of the Oxygen molecules. ans. 1045.13 J

102. Calculate the total internal energy of the Argon molecules. ans. 46.54 J

103. Calculate the total internal energy of the Carbon dioxide molecules. ans. 1.50 J

27 Work

We shall define the **thermodynamic work** done by, or upon, a system as it moves from an initial state to a final state by the line integral of pdV from the initial to the final state. **Work is dependent upon the process by which the system changes state and is therefore a process function**.

$$W = \int pdV \tag{27.1}$$

Using this definition of work, we can calculate the work done in each of the four common thermodynamic processes listed in table 6.1.[1]

Process	Work
Isothermal	$W = \int_{V_1}^{V^2} PdV = nRT \ln \frac{V_2}{V_1}$
Isobaric	$W = \int_{V_1}^{V^2} PdV = P(V_2 - V_1)$
Isochoric	$W = \int_{V_1}^{V^2} PdV = 0$
Adiabatic	$W = \int_{V_1}^{V^2} PdV = \frac{P_2V_2 - P_1V_1}{1-\gamma}$

Table 6.2: Work performed during thermodynamic processes.

Problems

104. Compute each of the formulas in table 6.2. Locate and plot each point on a PV diagram for the following problems.

105. One mole of a gas occupies a volume of 22.40 liters at a temperature of 293.00 degrees K. Calculate the pressure and total internal energy of the gas at this point (point 1). ans. 1.07 atm, 3654.21 J

106. Assume that $\gamma = 1.400$ for the gas and calculate the work done in an adiabatic expansion to 100 liters (point 2), the final pressure and final temperature at this point. ans. 2742.43 J, 0.1322 atm, 161.12 K

107. Calculate the work done in an isochoric expansion to a pressure of 2.000 atm (point 3) and temperature at this point. ans. 0 J, 2437.41 K

[1] For the adiabatic process, use must be made of the adiabatic rule $PV^{\gamma} = K$ where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume.

108. Calculate the work done in an isothermal compression back to a volume of 22.4 liters (point 4) and the pressure at this point. ans. -30,319.71 J, 8.93 atm

109. Calculate the total work done in an isochoric decompression back to 1.073 atm and the final temperature. 0 J, 293.00 K.

110. Plot each point on a PV diagram for the preceding problems.

28 Heat and Heat Absorption

We will define **heat** as energy in transit between a body and its surroundings. The transfer of energy from a system to its surroundings, or from its surroundings to a body, can be written as proportional to its rise in temperature.

$$\Delta Q = nC\Delta T \tag{28.1}$$

In this formula, $C = C_p$ is the **molar heat capacity at constant pressure** or $C = C_v$ is the **molar heat capacity at constant volume**. The number n represents the number of moles of the substance. Specific heat capacities are listed for most elements in Appendix V. The heat capacity of elements is often defined in terms of the **specific heat** of the element, which is defined as the *heat required to raise the temperature of 1 gram of the substance 1 degree centigrade*. The heat capacity is the product of the specific heat times the gram molecular weight.

Specific heat is often listed in different units which can lead to confusion. The units commonly used are $Joules/gm/^oC$, $calories/gm/^oC$ and $Btu/lb/^oF$. One calorie is equivalent to 4.189 Joules, and the calorie and Btu are defined such that

$$1\,\frac{calorie}{gm^oC} = 1\,\frac{kcal}{kg^oC} = 1\,\frac{Btu}{lb^oF}, \tag{28.2}$$

a rule which is often convenient to remember.

Significantly more heat is usually absorbed, or released, when a system changes phases.

$$\Delta Q = mH_f \tag{28.3}$$

$$\Delta Q = mH_v \tag{28.4}$$

The quantity H is the heat of fusion H_f when the system goes from solid to liquid or liquid to solid, and the heat of vaporization H_v when the system goes from

liquid to gas or gas to liquid. Since the amount of heat absorbed or released may depend on the path, heat is considered a process variable.

Example

When 100 grams of water, with the approximate heat capacities and latent heats listed in table **??** is heated at constant pressure from -20 to $+120$ degrees Centigrade, the heat energy absorbed is 74,000 calories.

ΔT	Process	Formula	$C_p \frac{cal}{g^oC}$	$H \frac{cal}{g}$	Calories
-20 to 0	Increase Temperature	$mC_p\Delta T$	0.5		1000
0	Melting	mH_f		80	8000
0 to 100	Increase Temperature	$mC_p\Delta T$	1.0		10000
100	Vaporization	mH_v		540	54000
100 to 120	Increase Temperature	$mC_p\Delta T$	0.5		1000
				Total	74000

Table 6.3: Absorption of heat as water is raised from -20 to 120 degrees-C.

28.1 Heat Transfer by Conduction

We can express the rate at which energy is conducted through a barrier as proportional to the area of the barrier and the differences in temperature on either side of the barrier and inversely proportional to the thickness of the barrier.

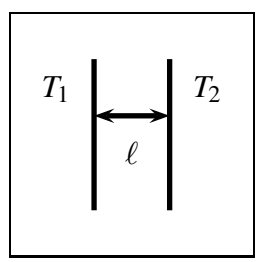


Figure 6.2: Heat conduction through a slab.

$$\frac{dQ}{dt} = \frac{kA}{\ell}(T_2 - T_1) \tag{28.5}$$

where A is the area, ℓ is the thickness of the barrier and T is the temperature on either side. In this formula, k is the **heat conduction coefficient**. In most scientific work, the heat conductivity per unit area k is expressed in $Watts/Meter/deg/K$. In engineering work, k is often expressed in $Btu/hour/foot/deg/F$. Conduction coefficients for several common materials are listed in Appendix G. Note should be taken that heat, internal energy

and work all have the same units. Because of the different systems of units, it is important to recall the useful conversion factors:[2]

$$1\frac{watt}{m\,^oK} = 0.002389\frac{calories}{cm\,^oC\,sec} = 0.5777\frac{Btu}{hr\,ft\,^oF} \tag{28.6}$$

Thermal conductivities vary with temperature, in some cases increasing and in some cases decreasing. For example, Type 304 Stainless steel will increase from about 8.3 at 20 degrees C to 12.5 $Watts/(meter\,deg\,C)$ at 650 degrees C while Carbon steel will decrease from about 43 at 20 degrees C to 25 $Watts/(meter\,deg\,C)$ at 650 degrees C

Example

The temperature differential across a glass window of 2 square meters area and 5 mm thickness is 20 degrees C. From Appendix G, the thermal conductivity of glass is 1.05 Watts/meter/C, so the rate at which heat is conducted through the window is

$$\frac{dQ}{dt} = \frac{(1.05)(2)(20)}{0.005} = 8400 \quad \text{watts} \tag{28.7}$$

28.2 Conduction through multiple layers

In the case of heat conduction through multiple layers the rate of heat conduction $\frac{dQ}{dt}$ must be the same through each layer.

This allows us to write

$$\frac{dQ}{dt} = k_1 A\frac{T_1 - T_2}{\ell_1} = k_2 A\frac{T_3 - T_2}{\ell_2}. \tag{28.8}$$

Canceling out T_2 gives

$$\frac{dQ}{dt} = A\frac{T_1 - T_3}{(\ell_1/k_1) + (\ell_2/k_2)}. \tag{28.9}$$

Thus the effective heat conduction coefficient is

$$k_{eff} = \frac{k_1 k_2}{k_2\ell_1 + k_1\ell_2}. \tag{28.10}$$

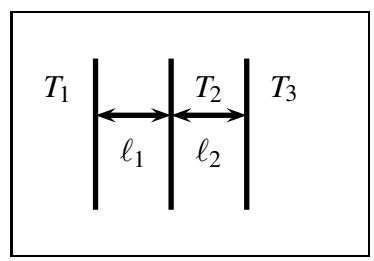


Figure 6.3: Heat conduction through multiple layers.

[2]The unit is properly interpreted to mean the heat transmitted through an area of 1 m^2 when the temperature gradient is 1 degree per meter. The units are expressed in different ways in different texts.

Using the effective heat conduction coefficient, the heat conduction rate could be calculated from

$$\frac{dQ}{dt} = Ak_{eff}(T_1 - T_3) \tag{28.11}$$

Example

A box built of hardwood with sides of 2 square meters and of thickness 1 cm thickness is lined with a 2 cm layer of polyurethane foam. What is the rate of heat conduction through the wall before and after lining with polyurethane if the temperature outside the box is 27 degrees C and inside is 47 degrees C? From Appendix G, the heat conduction coefficient of hardwood is 0.16 W/m/K and that of polyurethane is 0.02 W/m/K; so that the rate at which heat would be conducted through the hardwood side if it were unlined is

$$\frac{dQ}{dt} = \frac{(0.16)(2)(20)}{0.01} = 640 \quad \text{watts.} \tag{28.12}$$

After the 2 cm layer of polyurethane is added, the rate of heat conduction through the lined wall becomes using equation 28.9

$$\frac{dQ}{dt} = \frac{(2)(20)}{\frac{0.01}{0.16} + \frac{0.02}{0.02}} = 37.8 \quad \text{watts} \tag{28.13}$$

implying that the layer of polyurethane substantially reduces the rate of heat conduction. Using equation 28.10 the effective heat conduction coefficient can be calculated,

$$k_{eff} = \frac{(0.16)(0.02)}{(0.02)(0.01) + (0.16)(0.02)} = 0.941 \quad watts/m^2/^oK, \tag{28.14}$$

and used to obtain the same result for the rate of heat conduction

$$\frac{dQ}{dt} = (0.941)(2)(20) = 37.8 \quad \text{watts.} \tag{28.15}$$

28.3 Heat conduction through cylindrical walls

Suppose that a cylindrical pipe of radius r_1 conducting liquid at temperature T_1 is surrounded by a cylindrical insulator to a radius r_2 and thickness $\ell = r_2 - r_1$ where it is in contact with the environment at a temperature T_2. In this case heat will be conducted through the insulator to the outside environment at a constant rate $\dot{Q}$.

We can solve equation 28.5 for the temperature and integrate between the inside and outside limits using $2\pi r dr \ell$ for a differential volume element of thickness dr to obtain an equation for the temperature differential and hence the heat flow rate.

$$\frac{dQ}{dt} = 2\pi k r \ell \frac{dT}{dr} \tag{28.16}$$

$$\int_{T_1}^{T^2} dT = \frac{\dot{Q}}{2\pi k \ell} \int_{r_1}^{r^2} \frac{dr}{r} \tag{28.17}$$

$$T_1 - T_2 = \frac{\dot{Q}}{2\pi k \ell} \ln \frac{r_2}{r_1} \tag{28.18}$$

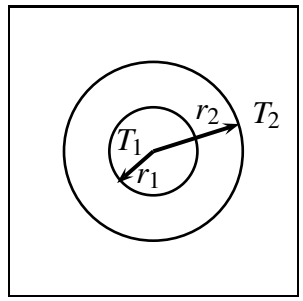


Figure 6.4: Heat conduction through cylindrical shell.

which provides an expression for the heat conduction rate

$$\frac{dQ}{dt} = \frac{2\pi k \ell (T_1 - T_2)}{\ln \frac{r_2}{r_1}} \tag{28.19}$$

If there are two concentric cylinders, as illustrated in figure 6.5, we follow the same procedure as before noting that the heat conduction rate through both cylinders must be the same and set the heat conduction rates equal.

$$\dot{Q} = \frac{2\pi k_1 \ell (T_1 - T_2)}{\ln \frac{r_2}{r_1}} = \frac{2\pi k_2 \ell (T_1 - T_3)}{\ln \frac{r_3}{r_2}} \tag{28.20}$$

where we have taken T_2 as the temperature at the interface between the two cylinders. Then solving for T_2 in one equation and inserting it in the other, we obtain an expression for the heat conduction rate independent of T_2,

$$\frac{dQ}{dt} = \frac{2\pi \ell}{\frac{1}{k_2} \ln \frac{r_3}{r_2} + \frac{1}{k_1} \ln \frac{r_2}{r_1}} (T_1 - T_3) \tag{28.21}$$

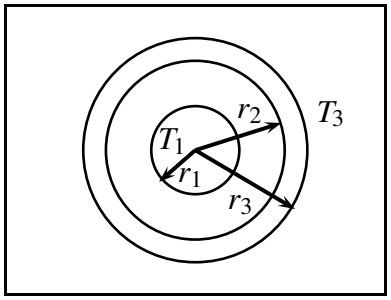


Figure 6.5: Heat conduction through two concentric cylindrical shells.

Example

Consider as an example a cast iron pipe 4 m in length conducting a liquid at a temperature 227 degrees C surrounded by an insulating shroud of asbestos which separates the pipe from the environment at 27 degrees C. Take the

outside diameter of the cast iron pipe to be 21 cm and wall thickness to be 1 cm with a thermal conductivity of 52 W/m/C and assume the asbestos shroud is 8 cm thick with a thermal conductivity of 0.100 W/m/C. To compute the heat loss from the pipe, we can substitute the given data into equation 28.21 to find a total heat loss rate of 1557 Watts.

$$\frac{dQ}{dt} = \frac{(6.2832)(4)}{\frac{1}{52}\ln\frac{29}{21} + \frac{1}{0.100}\ln\frac{21}{20}}(27-227) = 1557 \quad \text{watts} \tag{28.22}$$

28.4 Heat conduction through a spherical shell

Suppose that a reactor generates heat at a constant rate $\dot{Q}$ inside a spherical pressure vessel of inside radius r_1 and outside radius r_2. In this case we can follow the same procedure used for cylindrical pipes and integrate using a spherical volume element $4\pi r^2 dr$

$$\frac{dQ}{dt} = 4\pi k r^2 \ell \frac{dT}{dr} \tag{28.23}$$

$$\int_{T_1}^{T^2} dT = \frac{\dot{Q}}{4\pi k}\int_{r_1}^{r^2}\frac{dr}{r^2} \tag{28.24}$$

$$T_1 - T_2 = \frac{\dot{Q}}{4\pi k}\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \tag{28.25}$$

Then solving for the heat conduction rate we obtain

$$\frac{dQ}{dt} = 4\pi k \frac{r_1 r_2}{r_2 - r_1}(T_1 - T_2) \tag{28.26}$$

Example

Suppose that a reactor is encased in a hollow carbon steel sphere having inside radius of 20 cm and outside radius of 30 cm. If the reactor generates heat at the rate of 100 kW and the outside temperature of the sphere is maintained at 100 degrees C, what will be the equilibrium temperature of the inside surface of the sphere? Selecting the thermal conductivity for carbon steel from table G.1 and evaluating the constant factor of equation 28.26 for the sphere

$$R = \frac{(4)(3.1416)(43)(0.20)(0.30)}{0.30 - 0.20)} = 324 \quad watts/^oC \tag{28.27}$$

and solving equation 28.19 for the inside temperature T_1 we find

$$T_1 = \frac{RT_2 + Q}{R} = 408 \quad {}^oC \tag{28.28}$$

Problems

111. Mercury has a melting point of -39 ^oC, boiling point of 357 ^oC, latent heat of fusion 2.7 $calories/gram$ and latent heat of vaporization of 70.5 $calories/gram$ and specific heat of $0.033 calories/gram^oC$. Calculate the heat required to raise 100 g of Mercury at -75^oC to vapor at 450^oC. ans. 9,053 cal.

112. At what rate will a brick wall 3 inches thick, 20 feet long and 8 feet high conduct heat when the temperature difference is 40 degrees F? The thermal conductivity of fire clay brick is 0.81 $Btu/ft/hr/^oF$. ans. 20,736 Btu/hr.

113. What will the rate of heat conduction be if the brick wall is insulated with a one-half inch thick asbestos mill board having a heat conduction coefficient of 0.081 $Btu/ft/hr/^oF$? ans. 7,776 Btu/hr.

114. At what rate is heat lost from a copper pipe 6 meters long with a 15.0 cm inside diameter and wall thickness 3.00 cm when it is carrying heated water at 100 ^oC and the ambient temperature outside the pipe is 20.0 ^oC. The thermal conductivity of copper is 401 $watts/meter/^oC$. ans. 6.63 MW

115. At what rate will heat be lost after the copper pipe is insulated with a 15 cm layer of asbestos? The thermal conductivity of asbestos is 2.07 $watts/meter/^oC$. ans. 10.3 kW.

29 Heat transfer by radiation

All bodies radiate energy in the form of electromagnetic radiation. This law, known as the **Stefan-Boltzman Law**, was deduced by Josef Stefan (1835 – 1893) in 1879 on the basis of experimental measurements made by John Tyndall and was derived from theoretical considerations, using thermodynamics, by Ludwig Boltzmann (1844 – 1906) in 1884. Stated mathematically, the Stefan-Boltzman

law for emission of radiation from a body of surface area A is

$$P = \varepsilon A \sigma T^4 \quad \text{where} \tag{29.1}$$

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3} = 5.670400 \times 10^{-8}\ (Joules/meter^2\ sec\ K^4). \tag{29.2}$$

The constant σ is known as the Stefan-Boltzman constant and the constant ε is known as the **emissivity** of the body. In the formula for σ, **k** is the **Boltzmann** constant, **h** is Planck's constant, and **c** is the speed of light in a vacuum.

When radiant energy is incident upon a body, the radiant energy may be absorbed, reflected or transmitted. Any body which absorbs all of the radiation incident on it is referred to as a **blackbody** with an emissivity $\varepsilon = 1$.[3]

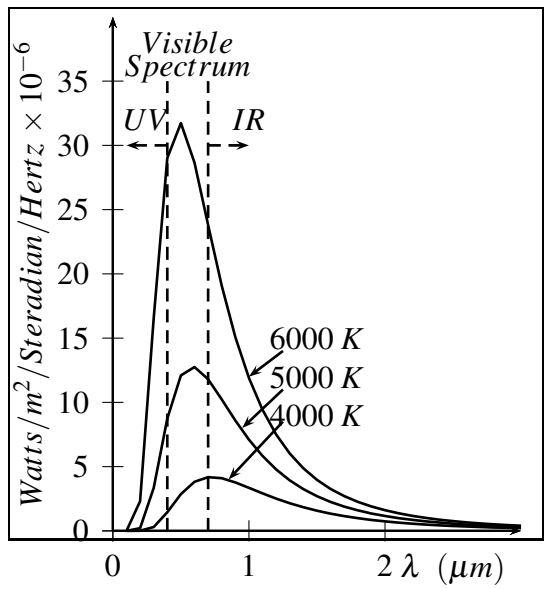


Figure 6.6: Radiation spectrum from bodies at 6000, 5000 and 4000 degrees K.

The name black body does not refer to the color of the body. A good example of a black body is a box with a small pinhole in it. All the radiation falling on the hole is absorbed by the box and the chances of that radiation being reflected a sufficient number of times to escape through the pinhole is negligible. Kirchoff's law states that the emissivity is equal to the absorptivity for any body at thermal equilibrium with its surroundings. If the body does not absorb all the radiation incident upon it, its emissivity will be less than unity.

The spectral distribution of black body radiation was published in 1901 by Max Planck (1858 – 1947) a German physicist who was awarded the Nobel Prize in Physics in 1918.

$$I(\lambda, T) d\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} d\lambda \tag{29.3}$$

In this formula, $I(\lambda, T)$ is the energy radiated per unit surface area per unit time per unit solid angle into the range of wavelengths between λ and $\lambda + d\lambda$ and has

[3]The term "blackbody", or "black-body" was introduced in 1860 by Gustav Kirchhoff.

units of $Watts/meter^2/d\Omega/d\lambda$. The wavelength at which peak emission occurs is obtained by differentiating equation 29.3, setting the result equal zero and solving for λ_{max}. The result, known as **Wein's Displacement Law** is

$$\lambda_{max} = \frac{0.0029}{T} \tag{29.4}$$

Several important physical phenomena can be addressed using these formulas.

1. Figure 6.6 illustrates the spectral distribution of radiation from a body at elevated temperatures 6000, 5000 and 4000 degress K and the range of wavelengths classified as ultraviolet, visible and infrared. It is clear that a considerable fraction of the radiation is emitted outside the visible spectrum at these temperatures and is unseen by the human eye.

2. The spectral distribution for 6000 K closely resembles the radiation spectrum from the Sun. The best fit occurs for a temperature of 5777 K, meaning that a black body radiation spectrum at 5777 K closely approximates that of the sun.

3. The Stefan-Boltzman law can be used to estimate the energy radiated by any object, including the human body. Take the average surface area of an adult male as 2 m^2 and the average temperature to be about 30 degrees C, or 303 degrees K while the ambient temperature is 293 degrees K. Then, from the Stefan-Boltzman law

$$P = \sigma\varepsilon A\left(T^4 - T_o^4\right) = 120\,Watts \tag{29.5}$$

   This amounts to a heat loss of 2477 kcal/day, or in terms of the medical definition of the word "calorie" 2477 calories per day.[4] Although this calculation does not take account of convection and evaporation and other factors, it is apparent that a significant fraction of the daily intake of calories by humans can be lost through thermal radiation.

4. Almost all the radiation emitted by the human body, or any other body at room temperature, and from a body at the temperature of boiling water is emitted in the infra red region as illustrated in figure 6.7. This explains why infrared sensors can pick up human bodies and other heated bodies, such as

[4] One kilocalorie = 1 calorie in medical terms

hot engines etc., at night when there is no sunlight to illuminate these bodies by reflection. Some reptiles, such as Rattlesnakes and other pit vipers, use their eyes to see during the day and infrared sensory organs at night to detect and hunt warm-blooded prey. Certain birds can also see to some extent in the ultraviolet range.

Another useful application of the Stefan-Boltzman law is the estimation of a planet's temperature. If we consider a planet with no atmosphere and no internal heat sources such as radioactive decay, its temperature will depend upon energy absorbed from its Sun less the energy it reflects less the energy it radiates. At equilibrium these three factors will add to zero.

$$\begin{aligned} \sigma\varepsilon A_e T_e^4 &= (1-\alpha)\,\sigma\varepsilon_s A_s T_s^4 \frac{\pi R_e^2}{4\pi D^2} \\ T_e &= T_s\sqrt{\frac{R_s}{2D}}\sqrt[4]{\frac{1-\alpha}{\varepsilon}} \qquad (29.6) \end{aligned}$$

Is is seen from equation 29.6 that the temperature of the Earth depends on temperature of the Sun multiplied by a constant. Taking the value of the parameters from table 6.4 and inserting in equation 29.6, we find the temperature that the Earth would have under these assumptions to be 258 K, or $-15\,C$.

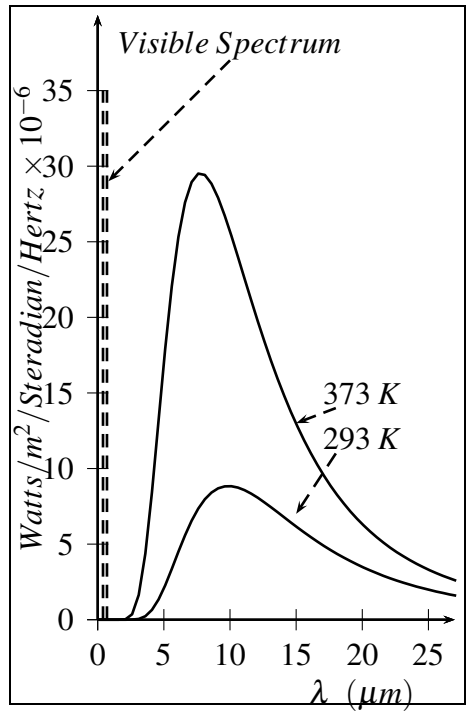


Figure 6.7: Radiation spectrum from bodies at room temperature and temperature of boiling water.

Since the temperature in caves is relatively uniform at 59 F or 274 K, we would expect this higher temperature to be due to heat generation inside the Earth and the effect of the Earth's atmosphere in containing heat that would be irradiated away. Heat radiated from a black body at 273 K would be about 315 $Watts/m^2$. Heat generation inside the earth is believed largely due to radioactive decay and heat flow to the earth's surface is estimated to be less than 0.1 $Watt/m^2$ Therefore the containment of heat must be largely due to the atmosphere.

If the effect of the atmosphere in containing heat is reduced by about 20% the Earth's temperature would rise to the known temperature of 59 F. Also a rise of

Parameter	Value	Description
T_s	$5777\ K$	Surface Temperature of Sun
R_s	6.96×10^8	Radius of Sun
D	1.46×10^{11}	Distance from Earth to Sun
α	0.300	Reflectance of the Earth for Sunlight
ε	1	Emissivity of the Earth in infra red range

Table 6.4: Sun and Earth parameters

6% in the temperature of the Sun would be sufficient to raise the temperature of the earth to 59 F. A change of about 60% in the reflectivity of the Earth would be needed to achieve the same effect. One large sun spot 8×10^7 m in diameter at a nominal temperature of 4000 K could affect the total radiation output from the Sun into 4π steradians by only about 0.16%; but significantly more if the sunspot is directly between the sun and the earth.

Problems

116. Take the emissivity ε and reflectance α of the moon to be 0.95 and 0.12 respectively and estimate the surface temperature. ans. 3.55 C.

117. Using data in this section, calculate the watts/m^2 of radiation from the sun arriving at the earth. ans. 1440 watts/m^2

118. If the radiant energy reaching the surface of the earth after absorption and reflection by the atmosphere is 300 watts/m^2 and the overall efficiency of a solar panel and conversion system is 50%, what will be the area of a solar panel needed to supply 10 kW? ans. 67 m^2

119. Using Plank's law calculate and plot the energy emitted per unit wavelength for the temperatures 3,000 to 8,000 ^oK in steps of 1,000 ^oK is SI units.

30 Heat Convection

When heat is transferred to a fluid (gaseous or liquid) that is moving past a surface, the process is referred to as **heat convection**. Heat convection therefore involves mass transfer in contrast to heat transfer which does not involve movement of

mass. If the movement of mass is caused by natural forces such as gravity or variations in density and buoyancy the transfer of heat is called **natural convection**; if the movement of mass is caused or increased by pumps or other devices, the transfer of heat is called **forced convection**.

From a mathematical standpoint, convective heat transfer is described by an equation of the type

$$\frac{dQ}{dt} = hA\Delta T \tag{30.1}$$

Using this equation involves the concept of a **convective film**, which may be visualized as a thin layer of fluid next to the body from which the heat is being drawn. Under this concept, ΔT is the temperature difference between the hot body and the bulk of the fluid flowing past the hot body outside the convective film; A is the area of the hot body surface; and h is the **convective heat transfer coefficient** with units $Watts/area/degree$. It is not easy to tabulate values of convective heat transfer coefficients since the coefficient is a function of several variables including fluid properties and fluid flow characteristics. In the case of natural convection, typical convective heat transfer coefficients range from 5 to 25 $Watts/m^2K$ for air and 20 to 100 $Watts/m^2K$ for water. For the case of forced convection, typical convective heat transfer coefficients range from 10 to 200 $Watts/m^2K$ for air and 50 to $10,000$ $Watts/m^2K$ for water.

Another form of the convective heat transfer equation is used in the case of convective heat transfer through a wall that separates two fluids, as illustrated in figure 6.8. In this figure, temperature T_1 represents the temperature of a hot fluid flowing past the surface while T_4 represents the temperature of a cold fluid flowing past the opposite surface of a wall separating the two fluids. T_2 and T_3 represent the surface temperatures on either side of the barrier. h_{12} represents the convective heat transfer coefficient between the hot fluid and the barrier while h_{34} represents the convective heat transfer coefficient between the cold fluid and the barrier. The heat transfer coefficient of the barrier wall is represented by

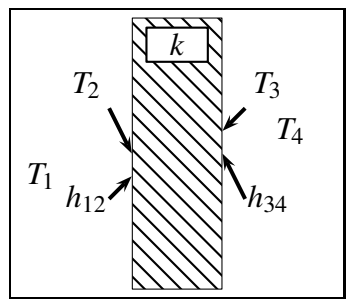


Figure 6.8: Heat convection through a partition.

k. In this case the heat transfer in each region is represented by a separate equation

$$\dot{Q} = Ak\left(\frac{T_2 - T_3}{x}\right) \tag{30.2}$$

$$\dot{Q} = Ah_{34}\left(T_3 - T_4\right) \tag{30.3}$$

$$\dot{Q} = Ah_{12}\left(T_1 - T_2\right) \tag{30.4}$$

The heat transfer rate through each layer must be the same. By solving each equation for the temperature difference and adding, we get

$$\dot{Q} = \frac{A\left(T_1 - T_4\right)}{\frac{1}{h_{12}} + \frac{x}{k} + \frac{1}{h_{34}}} \tag{30.5}$$

$$= AU\left(T_1 - T_4\right) \quad \text{where} \tag{30.6}$$

$$U = \frac{1}{h_{12}} + \frac{x}{k} + \frac{1}{h_{34}} \tag{30.7}$$

30.1 Example

Suppose that a hot liquid at a temperature of 500 F with a convection coefficient of 10 $Btu/hr/ft^2/F$ flows past a barrier 6-inches thick and area 50 sq-ft having a heat conduction coefficient of 24 $Btu/hr/ft/F$ with its opposite side in contact with air at an average room temperature of 100 F with a convection coefficient of 1.5 $Btu/hr/ft/F$. The rate of heat flow can be computed as follows

$$U = \frac{1}{10} + \frac{0.5}{24} + \frac{1}{1.5} = 1.27\ Btu/hr/ft/F \tag{30.8}$$

$$\dot{Q} = 25{,}381\ Btu/hr \tag{30.9}$$

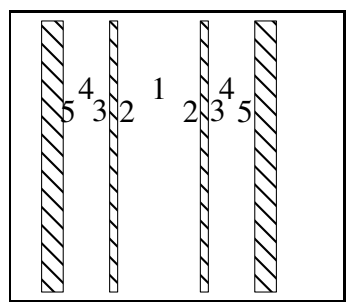


Figure 6.9: Heat convection through an encased pipe.

Consider another case in which a hot fluid is being transported through a pipe encased in a larger pipe through which a cold fluid is flowing, as illustrated in figure 6.9 where the surfaces are numbered from center to outside. We assume that there is no heat transfer through the outer pipe; take the heat conductivity of the inner pipe to be k; represent the convection coefficient at the inside of the inner pipe to be h_2 and at the outside of the inner pipe to be h_3 and designate the radii to the inner and outer surfaces

of the inner pipe to be r_2 and r_3 respectively. Following the same procedure and using equation 28.19 for the heat conduction through the wall of a cylindrical pipe, the equations for heat conduction and convection can be written as

$$\dot{Q} = k(2\pi L)\left(\frac{T_2 - T_3}{\ln\left(\frac{r_3}{r_2}\right)}\right) \tag{30.10}$$

$$\dot{Q} = h_{12}(2\pi r_2 L)(T_1 - T_2) \tag{30.11}$$

$$\dot{Q} = h_{34}(2\pi r_3 L)(T_3 - T_4) \tag{30.12}$$

As before, the rate of heat conduction must be the same through each layer. By solving each equation for the temperature difference and adding, we get

$$\dot{Q} = \frac{(T_1 - T_4)}{\frac{1}{2\pi r_2 h_{12}} + \left(\frac{1}{2\pi k}\right)\ln\frac{r_3}{r_2} + \frac{1}{2\pi r_3 h_{34}}} \tag{30.13}$$

$$= U(T_1 - T_4) \quad \text{where} \tag{30.14}$$

$$U = \frac{1}{\frac{1}{2\pi r_2 h_{12}} + \left(\frac{1}{2\pi k}\right)\ln\frac{r_3}{r_2} + \frac{1}{2\pi r_3 h_{34}}} \tag{30.15}$$

$$\tag{30.16}$$

as the equation for heat transfer per unit length through the walls of a pipe.

Problems

120. A steel pipe is insulated with a layer of asbestos. Show that the rate of heat conduction through the walls of a pipe of length ℓ is

$$\dot{Q} = K_{eff}\ell(T_{in} - T_{out}) \quad watts \quad \text{where} \tag{30.17}$$

$$K_{eff} = \frac{1}{\frac{1}{2\pi h_1 R_1} + \frac{1}{2\pi k_1}\ln\frac{R_2}{R_1} + \frac{1}{2\pi k_2}\ln\frac{R_3}{R_2} + \frac{1}{2\pi h_3 R_3}} \quad \frac{watts}{m\,^oK} \tag{30.18}$$

where h_1 is the convection coefficient for the film on the inside of the steel pipe, h_2 is the convection coefficient for the film on the outside of the insulation, k_1 is the heat transfer coefficient for the steel pipe and k_2 is the heat transfer coefficient for the insulation.

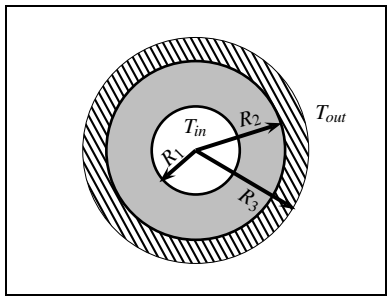


Figure 6.10: Heat conduction through insulated pipe.

121. Using the values h_1 = 50.0 $Btu/hr/ft^2/F$, h_2=5.00 $Btu/hr/ft^2/F$, k_1 = 20.00 Btu/hr/ft/F, k_2 = 0.150 Btu/hr/ft/F, R_1 = 3.00 in, R_2 = 4.00 in and R_3= 7.00 in calculate the overall heat transfer coefficient K. ans. 1.51 $Btu/hr/ft/F$

31 Laws of Thermodynamics

It is important to present clear statements of the laws of thermodynamics since the laws of thermodynamics are absolute physical laws and everything in the observable universe is subject to them. Drawing on the definitions preceding we are in a position to state these laws.

31.1 Zeroth Law of Thermodynamics

A system is said to be in thermal equilibrium with its surroundings when it experiences no net change in thermal energy. The zeroth law of thermodynamics states that **if system A is in thermal equilibrium with system B and system B is in thermal equilibrium with system C, then systems A and C are also in thermal equilibrium.** This law was understood for many years but not stated explicitly until British physicist Ralph H. Fowler first coined the term "zeroth law" in the 1920s based on a belief that it was more fundamental even than the other laws. Most scientists take the zeroth law to imply the existence of a thermodynamic variable which would be the same for all systems in equilibrium, or stated another way that the zeroth law empowers the construction of a thermometer. We now define that thermodynamic variable as temperature.

31.2 First Law of Thermodynamics

We have defined the internal energy of a system, the rate at which heat is transferred in or out of the system and the work done by or on the system by the following equations.[5]

$$U = \frac{3}{2}kT \tag{31.1}$$

$$\Delta Q = mC_i\Delta T \tag{31.2}$$

$$W = \int pdV \tag{31.3}$$

These definitions can be combined in a general statement of the conservation of energy known as the **First Law of Thermodynamics**. The first law of thermodynamics is not derived mathematically; it is a generalization of the results of many experiments. Simply stated, the first law says that **The change in the internal energy of a system equals the heat absorbed less the work done by the system**. Mathematically, this law is often expressed by:

$$\Delta U = Q - W \tag{31.4}$$

The first law may be used to establish useful relationships between the molar gas constant and the molar heat capacities for an ideal gas. Suppose for example that an ideal monoatomic gas is heated at constant pressure. In this case, the first law allows us to write

$$nC_v dT = nC_p dT - pdV \tag{31.5}$$

$$= nC_p dT - nRdT \tag{31.6}$$

Collecting terms and canceling the number of moles gives

$$R = C_p - C_v \tag{31.7}$$

Now, making use of the definition of internal energy for a monoatomic gas in equation 26.1, $U = \frac{3}{2}nRT$, we find

$$C_v = \frac{3}{2}R = 2.98\ calories \quad \text{and} \tag{31.8}$$

$$C_p = \frac{5}{2}R = 4.97\ calories \tag{31.9}$$

[5] In the formula $\Delta Q = mC_i\Delta T$, m is the mass and C_i is either C_p or C_v depending on the process.

When the gas is diatomic, there are two additional degrees of freedom so that the internal energy must be written as[6]

$$U = \frac{5}{2}nRT \tag{31.10}$$

and the molar heat capacities at constant pressure and constant volume become

$$C_v = \frac{5}{2}R = 4.97\,calories \tag{31.11}$$

$$C_p = \frac{7}{2}R = 6.95\,calories \tag{31.12}$$

It has been found that the molar heat capacities of monoatomic gases such as Argon and Helium and diatomic gases such as Oxygen, Nitrogen and Hydrogen are very close to these numbers. The molar heat capacities of polyatomic molecules will be greater than these values since the molecules possess additional degrees of freedom. The heat capacity of monoatomic gases is generally constant over a wide range of temperatures, but the heat capacity of diatomic and polyatomic gases generally increases with temperature and are often represented by a parabola of the type

$$C_p = a + bT + cT^2 \tag{31.13}$$

where b is typically 0.001a and c is typically 0.000000a.

In the applications that follow it is necessary to distinguish between open and closed systems. A **closed system** is one in which no transfer of mass can be made across its boundary while an **open system** is one in which mass can be transferred across the boundary.

Using the first law of thermodynamics we can supplement the ideal gas law with a relation between pressure and volume for any ideal gas undergoing a reversible adiabatic process in which $dQ = 0$. Starting with the first law for a reversible adiabatic process and using the ideal gas law, we can write

$$dU + PdV = 0 \tag{31.14}$$

$$C_v dT + nRT\frac{dV}{V} = 0 \tag{31.15}$$

$$C_v\frac{dT}{T} + nR\frac{dV}{V} = 0 \tag{31.16}$$

[6]The increase results from the change form 3 to 5 degrees of freedom for diatomic molecules and is embodied in the **equipartition theorem** from Statistical Mechanics which states that *for substances in equilibrium, there is an average energy of* $\frac{1}{2}nRT$ *associated with each degree of freedom.*

Integrating we obtain

$$C_v \ln T + nR \ln V = constant \tag{31.17}$$

Substituting γ for $\frac{C_p}{C_v}$ and using $R = C_p - C_v$, we can rearrange to get an expression that we will call the **adiabatic rule**.

$$\begin{aligned}
C_v \ln \frac{PV}{nR} + n(C_p - C_v)\ln V &= const && (31.18)\\
\ln \frac{PV}{nR} + n(\gamma - 1)\ln V &= const && (31.19)\\
\ln \frac{PV}{nR} + n \ln V^{(\gamma-1)} &= const && (31.20)\\
\ln \left(\frac{PV}{nR} n V^{(\gamma-1)} \right) &= const \quad \text{or} && (31.21)\\
PV^{\gamma} &= const && (31.22)
\end{aligned}$$

This rule is more useful expressed another way.

$$PV^{\gamma} = P_o V_o^{\gamma} \tag{31.23}$$

31.3 Second Law of Thermodynamics

The first law is a simple statement of the conservation of energy and it has never been violated. However, there is a problem in that it does not prohibit certain processes that we have found by experiment to be impossible–or at least they have never been accomplished. For example, an internal combustion engine is an excellent device for converting stored chemical energy into useful work. The first law expresses this possibility accurately and provides that the energy lost in conversion can be exhausted as heat energy. Internal combustion engines usually have efficiencies that range up to about 40% in the most efficient engines. The first law does not prohibit an internal combustion engine from being built with 100% efficiency; yet it has never been possible to build such an engine. As another example, suppose a block of steel at a high temperature is dropped into a container of cold liquid. As it is easy to demonstrate, the heat will flow from the hot steel to the cold liquid. There is nothing in the first law to prohibit the flow of heat back into the steel block, but it has never been observed. This begs the question *Are these things possible and we just simply cannot accomplish them?* or *Is there some other law of nature that prohibits these processes?* Once again, we are in the position of having to state an empirical law that we cannot derive from a more

fundamental law of nature but one of which we are confident as a result of many careful experiments. There are two statements of this law, either one of which can be derived from the other.

One is known as the **Clausis statement**, named for Rudolph Julius Emmanuel clausius (1822-1888) a German mathematical physicist, and stated in several ways. (1) *It is impossible for any device to operate in such a way as to produce no effect other than the transfer of heat from one body to another.* or (2) *Heat cannot by itself pass from a cold to a hot body.*

Another statement is known as the **Kevin-Plank statement**, named for William Thomson (Lord) Kelvin (1824-1907) who for 53 years was professor of natural science at the University of Glasgow and Max Plank (1858-1947) who was professor of physics at the University of Berlin. This statement may be expressed as *It is impossible for any device to operate in a cycle and produce work while exchanging heat only with bodies at a single fixed temperature.* This statement is equivalent in every way to that of Clausis since the violation of either statement will amount to a violation of the other.

31.4 Entropy

The first law led to the definition of internal energy. Early work was impeded by difficulties in defining "heat" and progress was made only after attention was directed toward how heat and work were related. In much the same way, the second law leads to the definition of **Entropy**, designated by the symbl S. As opposed to other thermodynamic variables that are based on physical quantities, entropy is defined only by a mathematical formula. It is difficult to give an exact definition of what entropy is, but we can define how it can be used.

$$\Delta S = \int_{rev} \frac{dQ}{T} \tag{31.24}$$

Mathematical proof of this law is obtained by an empirical evaluation of the integral $\int_{rev} \frac{dQ}{T}$ around any reversible, closed path and finding that its value is always zero. The vanishing of the line integral is sufficient and necessary cause to conclude that entropy, or the change in entropy, is a thermodynamic property of a system.

Changes in entropy may be calculated for an ideal gas which absorbs an amount of heat dQ in a reversible process.

Examples

As an example, we might want to calculate the change in entropy for 1 mole of Helium when it is heated reversibly at constant pressure from 101.3 kPa at 30 C to 150 C in a closed system. Making use of equations 31.1, the definition of ΔS can be rewritten for the purposes of calculation as

$$\Delta S = \int_1^2 \frac{mC_p dT}{T} = mC_p \ln \frac{T_2}{T_1} \tag{31.25}$$

Then using $C_p = C_v + R = \frac{7}{2}R$ for the molar heat capacity of Helium, and converting temperatures to Kelvin we obtain

$$\Delta S = \frac{7}{2}(8.31) \ln \frac{423}{303} = 9.70\, Joules/K \tag{31.26}$$

In this particular example, the initial pressure and temperature as well as the final pressure and temperature would be the same whether the system is open or closed. Therefore, the entropy change would be the same for an open system.

31.5 Third Law of Thermodynamics

The third law was developed by the chemist Walther Nernst between 1906-1912, and is sometimes referred to as Nernst's theorem or Nernst's postulate. The most widely accepted statement of the third law is that *It is impossible by any procedure, no matter how idealised, to reduce any system to the absolute zero of temperature in a finite number of operations.*; although other equally valid statements exist. Reference to a state known as "absolute zero" implies that there is a zero point of temperature at which all motion ceases to exist. This lowest possible temperature in the universe corresponds to about -273.15^o Celsius, or -459.7^o Fahrenheit and is defined as 0° Kelvin. Because of the definition of changes in entropy it is also implied that there is a zero point or lowest constant entropy.

Reaching absolute zero was an objective in many laboratories throughout the twentieth century. The lowest temperature reached by 1999 was 100 picokelvins by cooling a piece of rhodium metal. By November 2000, nuclear spin temperatures below 100 pK were reported for an experiment at the Helsinki University of Technology's Low Temperature Lab. Current work is being done to reach temperatures in the range of femtokelvins. The average temperature of the universe today is 2.73^oK, which is due to cosmic microwave background radiation. However, in February 2003, the Boomerang Nebula was found to have been releasing gases

at a speed of 500,000 km/h for the last 1,500 years which has cooled it down to approximately 1^oK.

Problems

122. Calculate the change in entropy of 1 kg of ice as it melts into water at constant temperature of 0 degrees C. The latent heat of fusion of water is 80 calories/g. ans. 293 cal/K

123. Calculate the change in entropy of 1 kg of water as it evaporates into steam at 100 degrees C. The latent heat of vaporization of water is 540 calories/g. ans. 1447 cal/K

124. A mass of 100 g of lead with $C_p = 0.0345$ cal/g/C at 100 degrees C is mixed with 200 g of water with C_p=1.00 cal/g/C at 20 degrees C. Calculate the resulting temperature of the mixture, the change in entropy of the lead shot, the water and the total system. ans. 21.36 degrees C, -0.816 cal/K, 0.926 cal/K, 0.11 cal/K

32 Enthalpy

Because the term $U + PV$ often appears in the analysis of systems, the **enthalpy** of a thermodynamic system is defined by the expression

$$H = U + PV \tag{32.1}$$

By differentiating this expression for the enthalpy and using the first law, we can show that

$$\begin{aligned} dH &= dU + PdV + VdP && (32.2)\\ &= dQ + VdP && (32.3)\\ &= TdS + VdP && (32.4) \end{aligned}$$

This reveals that the rate of change of enthalpy with respect to temperature is equivalent to the specific heat of the system at constant pressure.

$$\frac{dH}{dT} = nC_p \tag{32.5}$$

Combining the definitions of entropy and enthalpy allows the first law to be rewritten in a form that is useful in analyzing the behavior of water and steam. We can start by defining H as a function of S and P, that is $H = H(S,P)$, and differentiating.

$$dH = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP \tag{32.6}$$

Then using equations 32.4, we can establish the following relations

$$T = \left(\frac{\partial H}{\partial S}\right)_P \quad \text{and} \tag{32.7}$$

$$V = \left(\frac{\partial H}{\partial P}\right)_S . \tag{32.8}$$

At this point it is important to distinguish between enthalpy, internal energy and heat. Consider the case of 100 g of an ideal gas with a heat capacity of 0.248 cal/g/C being heated from $T_1 = 100^oC$ to $T_2 = 120^oC$ by the addition of heat at atmospheric pressure to produce a volume change from V_1 to V_2, the heat absorbed and changes in enthalpy as follows.

$$Q = mC_p\Delta T = (100)(0.248)(20) = 496 \quad \text{cal} \tag{32.9}$$

$$H = Q = 496 \quad \text{cal} \tag{32.10}$$

Since we have assumed an ideal gas we can use the ideal gas law to calculate the volumes V_1 and V_2.

$$V_1 = 1.0933 \quad m^3 \tag{32.11}$$

$$V_2 = 1.1519 \quad m^3 \tag{32.12}$$

The work done against the atmosphere in expansion and the change in internal energy of the gas can then be calculated.

$$W = P\Delta V = (1.013 \times 10^5)(0.0586) = 142 \quad \text{cal} \tag{32.13}$$

$$\Delta U = H - W = 354 \quad \text{cal} \tag{32.14}$$

Therefore, we may conclude that the enthalpy is a better measure of the energy absorbed than the internal energy, because some of the internal energy must be forfeited to perform the work of expansion.

These formulas and calculations apply in the case of an ideal gas in which distinctions are clear. Another case in which the ideal gas laws do not apply is in conversion of ice to steam as described in table 6.3, which is expanded here to show the changes in enthalpy, internal energy and the work done against the atmosphere during a process in which 100 g of water is taken from ice at -20^oC to steam at 120^oC by heating at constant pressure.

ΔT	Process	ΔQ	ΔH	ΔU	ΔW
-20 to 0	heating	1000	1000	1000	0
0	Melting	8000	8000	8000	0
0 to 100	heating	10000	10000	10000	0
100	Vaporization	54000	54000	52560	1440
100 to 120	heating	1000	1000	847	153
		74000	74000	71030	2970

Table 6.5: Changes in Q, H, U and W as water is raised from -20 to 120 degrees-C.

In the case of heat absorption, ΔQ, the calculation is straight forward application of the formula $\Delta Q = mC_p\Delta T$ in the heating phases, the formula $\Delta Q = mh_f$ for melting and $\Delta Q = mh_v$ for vaporization. Q is not a state variable but a process quantity which describes the absorption of heat depending on the process. The absorption of heat can be verified by direct measurement in the laboratory.

Enthalpy is a state function and an extensive quantity. It includes the internal energy stored in the water whether in an ice, water or steam state and the energy required to do work in pushing back the environment to accommodate the steam as the water vaporizes and as the steam expands when heated. It is equal to the absorption of heat in all steps.[7]

Internal energy includes both the kinetic energy of the molecules of water in the forms of ice, water or steam and the potential energy stored in binding the molecules together. Therefore it is not necessary that the temperature change when ice is melting or water is evaporating into steam. Also, the energy absorbed by ice and water in the processes of heating can be stored as both kinetic energy and potential energy so that it is not necessary to relate the temperature of the ice or water to a kinetic energy. For this reason, the changes in enthalpy and internal energy can be equal in the heating processes between -20 and 100 degrees C and

[7] Josiah Willard Gibbs introduced the concept of enthalpy in 1875 but the word enthalpy was first used in 1909 by J. P. Dalton, who accredited the name to Heike Kamerlingh Onnes.

the melting process. In these phases, the change in volume as heat is added is infinitesimal therefore the work done against the surroundings may be neglected.

However, the vapor released during vaporization must do work to push back the environment so that part of the heat absorbed must be used to do work thereby reducing the increase in internal energy. The same reasoning holds true in the heating process for vapor from 100 to 120 degrees C. The internal energy forfeited to doing work can be recovered when the system collapses back into its original state.

The work done pushing back the environment in vaporization and heating vapor can be calculated using the formulas $W = P\Delta V$ and equals the difference between the enthalpy change and the internal energy change. Obtaining the volume ΔV occupied by the steam may, however, be difficult. The ideal gas law does not work except at extremely high temperatures and/or low pressures in the superheated vapor domain. Errors encountered using the ideal gas equation near the critical point may exceed 300% but decrease as points are selected further down the saturation pressure curve. Therefore the volume occupied by the saturated vapor should be taken from validated steam tables. To calculate the work done against the atmosphere in table 6.5 the specific volume for saturated steam was taken from the steam tables for 100 g of steam to obtain 0.0603 m^3 at 100 degrees C and 0.0639 m^3 at 120 degrees C. Using these values, $W = P\Delta V$ gives 1440 calories for work required to push back the atmosphere in vaporization and 153 calories to heat the steam to 120 degrees.

These remarks may be concluded by observing that enthalpy is a better measure of the total energy of a system because it includes both the internal energy and the energy required to modify the surroundings to accommodate expansion of the system. The heat input to a system can be measured but it is difficult to measure the internal energy and the enthalpy. For this reason, we work with changes in these two quantities in engineering and scientific work. For example, in steam tables commonly used in engineering, the zero point of internal energy and entropy is usually set at zero at the triple point of water. At this point the enthalpy will have a small positive value.

Problems

125. The specific heat of Oxygen at constant pressure is 7.05 cal/mole/C. Calculate the change in enthalpy, work and internal energy done as the temperature of 2 moles of Oxygen is raised at atmospheric pressure from 20 ^oC to

100 ^oC. ans. 1128, 318 and 810 calories

126. Show that at constant pressure $\Delta U = \Delta H - P\Delta V$; at constant volume $\Delta H = \Delta U - V\Delta P$ and at constant temperature $\Delta U = \Delta H = 0$.

127. Calculate the heat absorbed and the changes in enthalpy and internal energy and the work done against the atmosphere in boiling 60 grams of water at atmospheric pressure. Take the heat of vaporization to be 540 calories/gm and the specific volume of dry steam at atmospheric pressure and 373 ^oK to be 26.8 ft^3/lb. ans. 32400, 32400, 29966 and 2434 calories

33 Thermodynamic Cycles

The thermodynamic cycles in the following problems are the basis for understanding engines of different types. A **thermodynamic cycle** consists of a series of thermodynamic processes transferring a system from one state to another and eventually returning the system to its initial state.

State variables depend only on the thermodynamic state and not the process by which the system is brought to that state. Therefore, the cumulative variation of such properties adds up to zero during a cycle. **Process quantities**, such as heat and work, depend on the path between two states and the cumulative variation adds up to be non-zero. However, the first law of thermodynamics applies not only to individual processes but to the cumulative process that is the cycle. Since internal energy is a state variable, it adds up to zero during any cycle so that the cumulative heat change equals the cumulative work, or stated another way, the net heat input is equal to the net work output over any cycle.

Two important classes of thermodynamic cycles are power cycles and heat pump cycles. **Power cycles** are cycles which convert some heat input into mechanical work. **Heat pump cycles** transfer heat from low to high temperatures using mechanical work input. In some cases thermodynamic cycles can be operated as either work producing cycles or heat pump cycles by controlling the process direction.

To begin our study of thermodynamic cycles, we need formulas for calculating the changes in thermodynamic variables during different processes.

33.1 Isothermal processes

Since the temperature remains constant in an isothermal process, the internal energy must also remain constant and the heat flow is equal the work done. The defining equations for an isothermal process are:

$$P_i V_i = P_f V_f = nRT \tag{33.1}$$

$$PdV + VdP = nRdT = 0 \tag{33.2}$$

$$PdV = -VdP \tag{33.3}$$

$$\left(\frac{P_i}{P_f}\right) = \left(\frac{V_f}{V_i}\right) \tag{33.4}$$

The changes in internal energy, heat absorbed or rejected, work done by or on the system, entropy and enthalpy are calculated for an isothermal process to be:

$$\Delta U = mC_v\left(T_f - T_i\right) = 0 \tag{33.5}$$

$$\Delta Q = W \tag{33.6}$$

$$W = nRT\ln\frac{V_f}{V_i} = nRT\ln\frac{P_i}{P_f} \tag{33.7}$$

$$\Delta S = nR\ln\frac{V_f}{V_i} = nR\ln\frac{P_i}{P_f} \tag{33.8}$$

$$\Delta H = nC_p\left(T_f - T_i\right) = 0 \tag{33.9}$$

33.2 Isochoric processes

Since the volume remains constant in an isochoric process, no work can be done on or by the system so that the internal energy must equal the heat flow. The defining equations for an isochoric process are:

$$PV = nRT \tag{33.10}$$

$$V_i = V_f \tag{33.11}$$

$$VdP = nRdT \tag{33.12}$$

$$\left(\frac{P_i}{P_f}\right) = \left(\frac{T_i}{T_f}\right) \tag{33.13}$$

The changes in internal energy, heat absorbed or rejected, work done by or on the system, entropy and enthalpy are calculated for an isochoric process to be:

$$\Delta U = nC_v \left(T_f - T_i\right) \tag{33.14}$$
$$\Delta Q = \Delta H \tag{33.15}$$
$$W = 0 \tag{33.16}$$
$$\Delta S = nC_v \ln \frac{T_f}{T_i} \tag{33.17}$$
$$\Delta H = nC_p \left(T_f - T_i\right) \tag{33.18}$$

33.3 Isobaric processes

Since the pressure remains constant in an isobaric process, the volume must change with changes in temperature and the heat flow will equal the change in enthalpy. The defining equations for an isochoric process are:

$$PV = nRT \tag{33.19}$$
$$P_i = P_f \tag{33.20}$$
$$PdV = nRdT \tag{33.21}$$
$$\left(\frac{V_i}{V_f}\right) = \left(\frac{T_i}{T_f}\right) \tag{33.22}$$

The changes in internal energy, heat absorbed or rejected, work done by or on the system, entropy and enthalpy are calculated for an isobaric process to be:

$$\Delta U = nC_v \left(T_f - T_i\right) \tag{33.23}$$
$$\Delta Q = \Delta H = nC_p \left(T_f - T_i\right) \tag{33.24}$$
$$W = P\left(V_f - V_i\right) = nR\left(T_f - T_i\right) \tag{33.25}$$
$$\Delta S = nC_p \ln \frac{T_f}{T_i} \tag{33.26}$$
$$\tag{33.27}$$

33.4 Adiabatic processes

Since no heat can be absorbed or rejected in an isobaric process, the change in internal energy must equal the negative work done on or by the system. The

defining equations for an adiabatic process are:

$$PV^{\gamma} = constant \tag{33.28a}$$

$$TV^{\gamma-1} = constant \tag{33.28b}$$

$$\left(\frac{P_i}{P_f}\right)^{\gamma} = \left(\frac{T_i}{T_f}\right)^{\gamma-1} \tag{33.28c}$$

The changes in internal energy, heat absorbed or rejected, work done by or on the system, entropy and enthalpy are calculated to be:

$$\Delta U = mC_v\left(T_f - T_i\right) \tag{33.29}$$

$$\Delta Q = 0 \tag{33.30}$$

$$W = \Delta U \tag{33.31}$$

$$= \frac{\left(P_iV_i - P_fV_f\right)}{\gamma - 1} \tag{33.32}$$

$$\Delta S = 0 \tag{33.33}$$

$$\Delta H = nC_p\left(T_f - T_i\right) \tag{33.34}$$

These formulas are partially summarized in table 6.6.

Process	Internal Energy	Heat	Work	Entropy	Enthalpy
Isobaric	$nC_v\Delta T$	$nC_p\Delta T$	$P\Delta V$	$nC_p \ln\frac{T_2}{T_1}$	$nC_p\Delta T$
Isochoric	$nC_v\Delta T$	$nC_v\Delta T$	0	$nC_v \ln\frac{T_2}{T_1}$	$nC_p\Delta T$
Isothermal	0	$nRT \ln\frac{V_2}{V_1}$	$nRT \ln\frac{V_2}{V_1}$	$nR \ln\frac{V_2}{V_1}$	0
Adiabatic	$nC_v\Delta T$	0	$-nC_V\Delta T$	0	$nC_p\Delta T$

Table 6.6: Changes in thermodynamic variables during thermodynamic processes.

We cannot calculate absolute values of the entropy, but changes in entropy serve equally well for studying thermodynamic processes. A good way to understand internal combustion engines is by an examination of the thermodynamic processes the engines utilize in performing work. We will analyze several engines in the following sections, beginning with the Carnot cycle.

We define the efficiency of a thermodynamic cycle in heat engines as the ratio of the net work output to the heat input as follows.

$$\eta = \frac{W_{net}}{Q_{input}} \tag{33.35}$$

By replacing the net work and heat input with specific formulas for the thermodynamic processes involved, we can reduce equation 33.35 to an equation involving only the state variables, temperature, pressure or volume.

Problems

128. Complete the following table for 2.000 moles of Hydrogen taking C_p = 6.900 cal/mole/C, C_v = 4.9137 cal/mole/C, γ = 1.40426 and R = 0.082076 l atm/deg/mole and treating Hydrogen as an ideal gas.

	Units	Isobaric	Isochoric	Isothermal	Adiabbaic
P1	atm	2.0	2.0	2.0	2.0
V1	liters	24.6	24.6	24.6	24.6
T1	Kelvin	300.0	300.0	300.0	300.0
P2	atm	2.0			2.7
V2	liters		24.6	32.82	
T2	Kelvin	400.0	400.0	300.0	
W	cal				
Q	cal				
H	cal				
U	cal				
S	cal				

34 Air Standard Cycles

By definition, an **internal combustion engine** is one in which the addition of heat occurs within the engine and results from combustion of a fuel to convert chemical energy into heat energy. The heat energy is subsequently converted into mechanical energy, or work, and the rate at which work is done is the power output of the engine. Since a gas in which the products of combustion are continuously increased cannot be reused as a working fluid in a closed system, the internal combustion engine must be an **open system** in which fresh air is continuously drawn into the engine and heat is rejected by expelling hot combustion gases to the atmosphere. To illustrate the principles of physics involved in the internal combustion engine, however, it is convenient for the engine to operate as a **closed system**

For this purpose, we must replace the process of internal combustion with a heat source from which heat is added and the exhaust process with a heat sink which absorbs the heat during the heat removal cycle. The working fluid used in these engines is air; therefore, these theoretical engines are often called **air-standard engines**. However, using these air-standard cycles for engineering design and analysis is impracticable since they involve considerable deviation from the actual processes that take place in an internal combustion engine. Also the idealized assumptions in the air-standard cycles often result in excessively high temperatures and pressures that could not be tolerated in an engine made from steel. The air-standard cycle does, however, permit an illustration of the principles of physics involved in engines and suffice to establish certain relations between thermodynamic variables that describe how one variable changes due to changes in another. It is for this reason that they are extensively studied in both physics and engineering.

34.1 Carnot Cycle

Every thermodynamic system exists in a particular state. If such a system is taken through a series of different states finally returning to its starting point, the system may, or may not, perform work on its surroundings. When the system performs work on its surroundings, it is acting as a heat engine to produce work. The system can be in either liquid, gas or solid state and may exist in any form. Examples used to illustrate the principle of converting one form of energy to work often include electric cells, rubber bands, soap films, and chemical solutions. The first question that arises is how efficient can a heat engine be for converting heat into useful work.

This question was addressed by Nicolas LÃĹonard Sadi Carnot in 1824 who developed a hypothetical heat engine known as the Carnot heat engine. Carnot's work was further studied by BenoÃŞt Paul âĂęmile Clapeyron in 1834 and later by Rudolf Clausius. [8]

A heat engine acts by transferring energy from a warm region to a cool region of space and, in the process, converting some of that energy to mechanical work. The cycle may also be reversed to transfer thermal energy from a cooler system to a warmer one, thereby acting as a refrigerator or heat pump rather than a heat engine. These studies lead to the facts that: (1) the efficiency of a heat engine is dependent on the temperatures at which heat is supplied and rejected to

[8] The concept of entropy emerged from these studies.

the surroundings and not the working substance and (2) the most efficient thermodynamic cycle possible was one in which all the heat supplied is supplied at one fixed temperature and all the heat rejected is rejected to a lower fixed temperature. We can think of the Carnot heat engine as a gas enclosed in a chamber with a movable piston so that the gas can be alternately compressed or allowed to expand as illustrated in 6.11.

A Carnot cycle involves (1) a working system, (2) a source of energy at a high temperature, (3) a heat sink at a low temperature, (4) a means of joining the system alternately to either a source of heat energy at high temperature T_H (hot reservoir) or a heat sink at a low temperature T_C (cold reservoir) and a surrounding environment that can either supply or absorb work. Carnot cycles are illustrated in the PV and TS diagrams of figure 6.12 for an ideal gas. The Carnot cycle consists of:

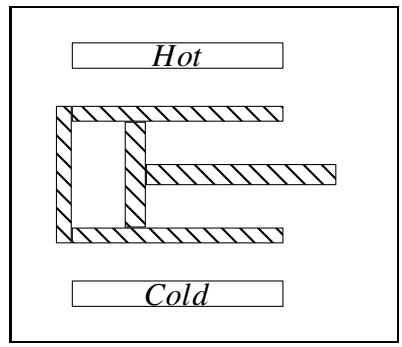


Figure 6.11: Illustrated Carnot cycle.

1. a reversible isothermal expansion of the gas at the high temperature from point (1) to point (2) during which heat is added from the hot reservoir and the piston made to do work on the surroundings.

2. a reversible adiabatic expansion of the gas from point (2) to point (3) during which the system is isolated from both the hot and cold reservoirs. Since the system is isolated, the expansion causes the gas to cool to a lower temperature; and since the gas is expanding and forcing the piston to move work is being done on the surroundings.

3. a reversible isothermal compression of the gas at the cold temperature from point (3) to point (4) during which the surroundings do work on the gas. The cold reservoir is brought into contact with the system so that heat is rejected to the low temperature reservoir.

4. a reversible adiabatic compression of the gas from point (4) back to point (1) during which the system is again isolated from both the hot and cold reservoirs. Since the system is isolated the compression raises the temperature to the high temperature and the surroundings do work on the system.

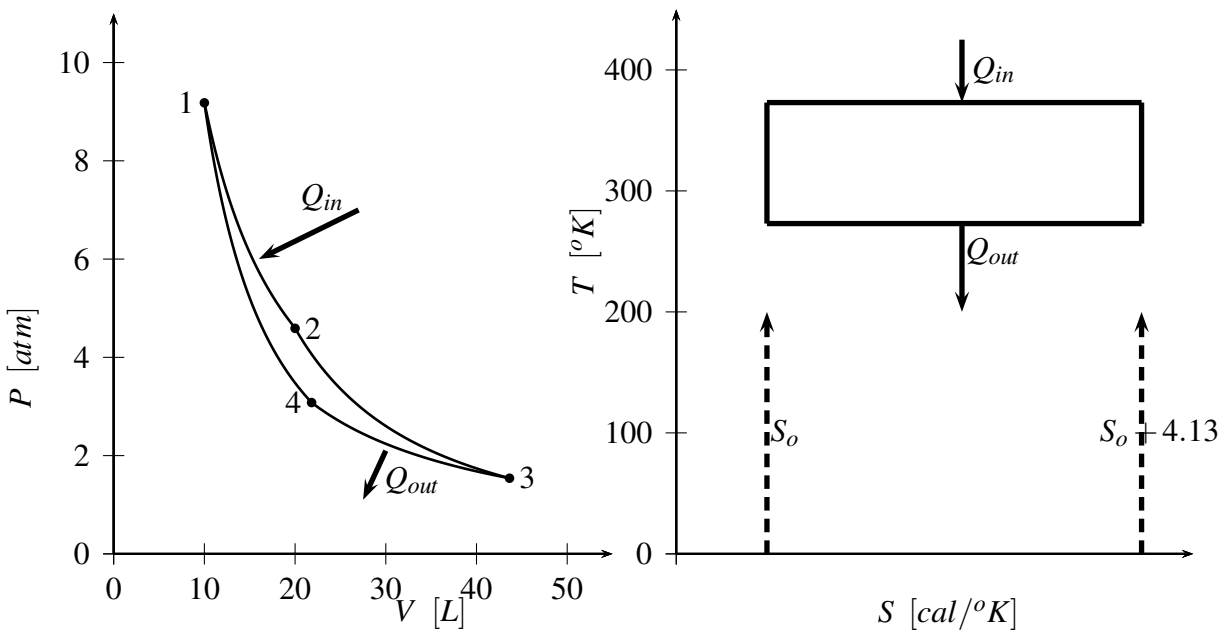


Figure 6.12: Carnot cycles on PV and TS diagrams.

We will take the working gas for the Carnot engine to consist of 2.00 moles of gas with $C_p = 6.95\ calories/mole^oK$ and $C_v = 4.96\ calories/mole^oK$ and the constant $\gamma = 1.40$. We also assume the working gas obeys the ideal gas law

$$PV = 0.082nT \tag{34.1}$$

and the adiabatic rule, which can be written as

$$P_1V_1^{1.4} = P_2V_2^{1.4} \tag{34.2}$$

To start the analysis, we will assume the Carnot cycle operates between the boiling temperature of water, 373 degrees K, and the freezing temperature of water, 273 degrees K. It is difficult to define pressure in an engine, so we will start with a volume of 10.00 liters and assume an isothermal expansion to a volume of 20.00 liters. The remainder of the points can be calculated using the ideal gas law and adiabatic rule.

$$P_1 = nRT_1/V_1 = (2)(0.082057(373)/10.00 = 9.18\ atm \tag{34.3}$$

At the point $V_2 = 20$ liters the ideal gas law can be used again to calculate P_2,

$$P_2 = nRT_2/V_2 = (2)(0.082057)(373)/20.00 = 4.59\ atm \tag{34.4}$$

The descent to pressure P_3 and volume V_3 by an adiabatic process requires use of the adiabatic rule, equation 33.28(b), to calculate V_3. Taking the logarithm and rearranging gives.

$$V_3 = 10^{\log V_2 + \frac{1}{\gamma-1}\log\left(\frac{T_2}{T_3}\right)} = 10^{\log 20.00 + \frac{1}{0.4}\log\left(\frac{373}{273}\right)} = 43.64\ liters \tag{34.5}$$

Using the ideal gas law to calculate P_3 gives

$$P_3 = nRT_3/V_3 = (2)(0.082057(273)/43.64 = 1.54\ atm. \tag{34.6}$$

The return path to point 1 will progress by an isothermal process to an intermediate point 4 and then by an adiabatic process back to point 1. The intermediate point V_4 must be calculated as if the cycle proceeded from point 1 to point 4 to obtain the intersection of the adiabatic process $4 \to 1$ with the isothermal process $3 \to 4$ for which the temperature is 300 degrees K.

$$V_4 = 10^{\log V_1 + \frac{1}{\gamma-1}\log\left(\frac{T_1}{T_4}\right)} = 10^{\log 10.00 + \frac{1}{0.4}\log\left(\frac{373}{273}\right)} = 21.82\ liters. \tag{34.7}$$

Then the pressure P_4 can be determined from the ideal gas law.

$$P_4 = nRT_4/V_4 = (2)(0.082057(300)/21.82 = 3.08\ atm \tag{34.8}$$

The calculated values are summarized in table 34.1. Now, the changes in internal energy, heat exchanged and work done can now be calculated using the formulas of table 6.6. To begin, the change in internal energy is always calculated from the defining equation 26.1 except when the process is isothermal in which case $\Delta U = 0$. It will be noted from these results that the internal energy changes only during the adiabatic processes increasing if the the temperature increases and decreasing if the temperature decreases. Also, it should be noted that the changes in the internal energy sum to zero, as they must since internal energy is a state variable.

$$\Delta U_{12} = nC_v\Delta T = 0 \tag{34.9}$$
$$\Delta U_{23} = nC_v\Delta T = (2)(4.96)(273-373) = -1488\ calories \tag{34.10}$$
$$\Delta U_{34} = nC_v\Delta T = 0 \tag{34.11}$$
$$\Delta U_{41} = nC_v\Delta T = (2)(4.96)(373-273) = 1488\ calories \tag{34.12}$$

The heat absorbed or rejected by the system in each process is calculated as follows. In this case, it will be noted that the heat inputs do not sum to zero. The

positive sum of 413 calories indicates that there is a net flow of heat energy into the system.

$$\Delta Q_{12} = nRT \ln \frac{V_2}{V_1} = +1540\, calories \tag{34.13}$$
$$\Delta Q_{23} = 0 \tag{34.14}$$
$$\Delta Q_{34} = nRT \ln \frac{V_4}{V_3} = -1127\, calories \tag{34.15}$$
$$\Delta Q_{41} = 0 \tag{34.16}$$

The next column is that of work calculated as follows. The cumulative sum of work done, 413 calories equals the cumulative sum of the heat absorbed as required by the first law. In this case the system does net work on the surroundings.

$$\Delta W_{12} = nRT \ln \frac{V_2}{V_1} = +1540\, calories \tag{34.17}$$
$$\Delta W_{23} = nC_v \Delta T = -(2)(4.96)(273-373) = +1488\, calories \tag{34.18}$$
$$\Delta W_{34} = nRT \ln \frac{V_4}{V_3} = -1127\, calories \tag{34.19}$$
$$\Delta W_{41} = nC_v \Delta T = -(2)(4.96)(373-273) = -1488\, calories \tag{34.20}$$

For the last column, the changes in entropy are calculated as follows. In this case the sum of the changes in entropy is zero.

$$\Delta S_{12} = \frac{\Delta Q_{12}}{T} = \frac{1540}{373} = +4.13\, calories/K \tag{34.21}$$
$$\Delta S_{23} = 0 \tag{34.22}$$
$$\Delta S_{34} = \frac{\Delta Q_{12}}{T} = \frac{-1127}{273} = -4.13\, calories/K \tag{34.23}$$
$$\Delta S_{41} = 0 \tag{34.24}$$

The TS diagram is obvious for the Carnot cycle. The lines and constant temperature are drawn at 273 and 373 degrees. We cannot define the entropy of the gas without a reference point, but the change in entropy is all that is needed. In each process, and for the total of all processes, the accuracy of calculations and correct signs can be checked by using the first law.

Using equation 33.35, we can calculate the efficiency of the Carnot cycle.

$$\eta = \frac{W_{net}}{Q_{input}} = \frac{413}{1540} = 0.268 \tag{34.25}$$

Point	P(atm)	V(liters)	T(^oK)
1	9.18	10.00	373
2	4.59	20.00	373
3	1.54	43.64	273
4	3.08	21.82	273

Process	ΔU	Q	W	ΔS
1-2	0	1540	1540	4.13
2-3	-1488	0	1488	0
3-4	0	-1127	-1127	4.13
4-1	1488	0	-1488	0
totals	0	413	413	0

Table 6.7: End point values of PVT with Work and Energy in the Carnot cycle

By careful selection of the starting points the efficiency of the Carnot engines we can obtain higher efficiencies, but never 100 % efficiency. In practice, efficiencies of practical engines will never be as high as that of the Carnot engine due to irreversibility in processes, etc.

Substituting the equations for the work output and the heat input reduces this definition to

$$\eta = 1 - \frac{T_C}{T_H} = 1 - \frac{273}{373} = 0.268 \tag{34.26}$$

which shows that the efficiency of the Carnot engine depends only upon the temperature of the hot and cold reservoirs and not on the properties of the gas used in the system.

The Carnot cycle provides an excellent example of how the second and third laws of thermodynamics reinforce the first law. Think of the Carnot cycle in terms of a larger amount of heat, 1540 calories, being absorbed from a heat source at higher temperature, 373 K, in process $1 \rightarrow 2$ and a smaller amount of heat, 1127 calories, being rejected to a heat sink at a lower temperature, 273 K in process $3 \rightarrow 4$. The difference, 413 calories, is then converted to work done by the system on its surroundings. Since the cycle is complete with the system being returned to its initial state, the change in internal energy, ΔU, is zero, in compliance with the first law of thermodynamics. The fact that the conversion of heat to work is not 100% efficient so that some heat must be lost is emphasized by the Clausis statement of the second law; and the fact that there must be a temperature difference between the heat source and heat sink is emphasized by the Plank statement of the second law. A careful examination of the Carnot cycle, and the many other cycles that have been studied, will also reveal that it is impossible to reduce the temperature of any system with a working fluid to absolute zero in a finite number of processes without violating the ideal gas law or the adiabatic rule as stated in the third law.

In order for work to be done there has to be a difference between the heat

absorbed and the heat rejected so that the sum of the heat exchanges during the cycle is not zero. However, the definition of entropy is such that the change in entropy is zero in a process where there is a net absorption or rejection of heat.

It may also be noted that the Carnot cycle operated in reverse would be a refrigerator in which work is done on the system by the surroundings to extract a smaller amount of heat from a reservoir at a lower temperature and rejecting a larger amount of heat to a reservoir at a higher temperature. As required by the Clausis statement, work must be done to achieve operation of the Carnot cycle in reverse. We owe the successful development of refrigeration systems to this principle of thermodynamics.

Once scientists understood the implications of these thermodynamic principles, a systematic study began on how to build more efficient and better engines that would convert heat to work. Contrary to popular opinion the process began in the early 1800s and still continues today. One of the first heat engines was invented in 1817 by the Reverend Robert Stirling (1790-1878), a Scottish minister, and modern jet and rocket engines are based on the same principles.

34.2 Otto Cycle

Modern internal combustion engines commonly use a four-stroke cycle. The four strokes refer to intake, compression, combustion (or power stroke), and exhaust strokes that occur during two crankshaft rotations per working cycle of the gasoline engine and diesel engine. The cycle begins at a point normally referred to as Top Dead Center (TDC). The strokes are described as follows:

1. Intake stroke–in which the piston descends from TDC to be bottom of the cylinder reducing pressure and allowing the fuel and air mixture to be forced in by atmospheric pressure, or greater in the case of supercharging. This stroke is an isobaric expansion process from point (0) to point (1).

2. Compression stroke–in which the piston returns to the top of the cylinder compressing the fuel and air mixture in an adiabatic compression process from point (1) to point (2).

3. Power stroke–in which the fuel and air mixture ignites due either to high pressure and/or a spark from a spark plug and builds up pressure in an isochoric heating process from point (2) to point (3), then driving the piston down to the bottom of the cylinder in an adiabatic expansion process from point (3) to point (4).

4. Exhaust stroke–in which the exhaust valve opens to release the compressed gas in an isochoric cooling process from point (4) to point (1) and the piston returns to TDC exhausting the remaining gas in an isobaric compression process from point (1) back to point (0).

The Otto cycle can be analyzed from a thermodynamic viewpoint by ignoring the isobaric expansion and the isobaric compression processes identified by the dashed lines between points (0) and (1) and calculating the changes in internal energy, heat absorption and rejection and work done by and on the system in the other processes.

For the analysis, we will assume a piston diameter of 9.216 cm and stroke of 15.00 cm to give a swept volume of 8.00 liters, a clearance volume of 15 %, or 1.20 liters, and a compression ratio of 7.67. We will also assume that the engine is fueled with an gasoline-air mixture at the stoichiometric ratio of 14.9 parts by weight of air to gasoline and an inlet temperature of 300 degrees K at atmospheric pressure. The heat of combustion of gasoline is $h = 10,607$ calories/gram, and the specific heat of air at constant pressure is 0.171 cal/g.

It should be pointed out that we also assume that the mass of air remains constant throughout the cycle, the specific heat of air is assumed constant, there is no dissociation of molecules and that heat is added at constant volume. We also ignore the effect of including gasoline in the adiabatic process from points (1) to (2) and the effects of its combustion in the isochoric process from points (2) to (3). None of these assumptions are true in an actual gasoline engine; therefore this analysis is at best an approximation in which both temperatures and pressures are higher than in practice. This raises the question of why this idealized study is undertaken. The answer is that making these idealized assumptions allows us to illustrate the physics of the thermodynamic cycle and heat engines without obscuring them in the details of an exact engineering analysis that takes account of all these factors.

Continuing with the analysis, we assign the values of $P_1 = 1.00$ atm, $V_1 = 1.20$ liters and $T_1 = 300$ degrees K. Then taking the fuel-air mixture to be an ideal gas so that we can use the ideal gas law, we find that the number of moles of fuel-air mixture that enter the engine during the intake stroke is

$$n = \frac{PV}{RT} = \frac{(1.00)(1.20)}{(0.082057)(300)} = 0.0467 moles \tag{34.27}$$

For the second point, we can use the adiabatic rule to calculate the pressure and

temperature.

$$P_2 = P_1 \left(\frac{V_1}{V_2}\right)^{\gamma} = (1.00)\left(\frac{1.15}{0.15}\right)^{1.4} = 17.32\,atm \tag{34.28}$$

$$T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\gamma-1} = (300)\left(\frac{1.15}{0.15}\right)^{0.4} = 678\,K \tag{34.29}$$

For the third point, we must have the heat admitted to the cylinder through combustion of the gasoline. Since air weighs 28.97 g/mole, the total mass of the fuel-air mixture admitted to the cylinder is $0.0467 \times 28.97 = 1.35$ g. Since the air/fuel ratio was taken to be 14.9 parts by weight, the amount of gasoline admitted to the cylinder is $1.35/14.9 = 0.0920$ g. Using the heat of combustion for gasoline $h = 10,607$ calories/gram, we can obtain the heat released in the cylinder.

$$\Delta Q = m_{fuel}h = (0.0920)(10607) = 976\,calories \tag{34.30}$$

Recall, that in the thermodynamic analysis of the air-standard Otto cycle we consider the heat input to come from a heat reservoir and not from internal combustion. To calculate the temperature at point (3), we therefore use the mass of air in the cylinder and the specific heat of air at constant volume.[9]

$$T_3 = \frac{\Delta Q}{m_{air}C_v} + T_2 = \frac{976}{(1.35)(0.171)} + 678 = 4883\,K \tag{34.31}$$

In this calculation, we have used the mass of air in the cylinder and the specific heat or air at constant volume. Then using the ideal gas law at points (3) and (2) and taking their ratio, we can obtain the pressure at point (3).

$$P_3 = P_2\left(\frac{T_3}{T_2}\right) = (17.32)\left(\frac{4883}{678}\right) = 124.8\,atm \tag{34.32}$$

The temperature and pressure at point (4) is then obtained using the adiabatic rule.

$$P_4 = P_3 \left(\frac{V_3}{V_4}\right)^{\gamma} = (124.8)\left(\frac{0.15}{1.15}\right)^{1.4} = 7.21\,atm \tag{34.33}$$

$$T_4 = T_3 \left(\frac{V_3}{V_4}\right)^{\gamma-1} = (4883)\left(\frac{0.15}{1.15}\right)^{0.4} = 2162\,K \tag{34.34}$$

[9] For the air-standard Otto cyle, $\Delta Q = m_{fuel}h = m_{air}C_v(T_3 - T_2)$. An accurate engineering analysis would include the molecular weight and mass of the fuel and the effect of the products of combustion.

Now the changes in internal energy, the heat absorbed and released and the work done by and on the system can be calculated. First the change in internal energy is calculated from the definition $m_{air}C_v\Delta T$ for each process.

$$\Delta U_{12} = (1.35)(0.171)(678-300) = 88\ calories \tag{34.35}$$
$$\Delta U_{23} = (1.35)(0.171)(4883-678) = 976\ calories \tag{34.36}$$
$$\Delta U_{34} = (1.35)(0.171)(2162-4883) = -632\ calories \tag{34.37}$$
$$\Delta U_{41} = (1.35)(0.171)(300-2162) = -432\ calories \tag{34.38}$$

It may be noted that the changes in internal energy sum to zero for the thermodynamic process.

$$\sum \Delta U_i = 88+976-632-432 = 0 \tag{34.39}$$

which provides a check on the calculations. Since the work done in the isochoric process from point (4) to (1) is zero, the heat released to the surroundings will, by the first law, equal the change in internal energy. This, however, assumes a slow conduction of heat through the cylinder walls and not the release of the gas through the exhaust valve.

The work done in the adiabatic process from point (3) to (4) will, again by the first law, equal the negative of the change in internal energy. The work may, however, be calculated from

$$W_{34} = \frac{P_4V_4 - P_3V_3}{0.4} = -26.1\ liter-atm = -632\ calories \tag{34.40}$$

which provides a check on use of the equivalence of work and changes in the internal energy for the value. Finally, we are in a position to calculate the efficiency of the engine.

$$\eta = 1 - \frac{T_3 - T_2}{T_4 - T_1} = 56\ \% \tag{34.41}$$

For sake of completeness, the change in entropy is also calculated for each of the isochoric processes and found to be be equal but opposite in sign.

$$\Delta S_{23} = mC_v \ln\frac{T_3}{T_2} = (1.35)(0.171)ln\frac{4883}{678} = 0.46\ calories/degree \tag{34.42}$$
$$\Delta S_{41} = mC_v \ln\frac{T_1}{T_4} = (1.35)(0.171)ln\frac{300}{2162} = -0.46\ calories/degree \tag{34.43}$$

Finally, we are in a position to calculate the efficiency of the Otto cycle from the definition of efficiency, equation 33.35.

$$\eta = \frac{W_{net}}{Q_{input}} = \frac{544}{976} = 0.557 \tag{34.44}$$

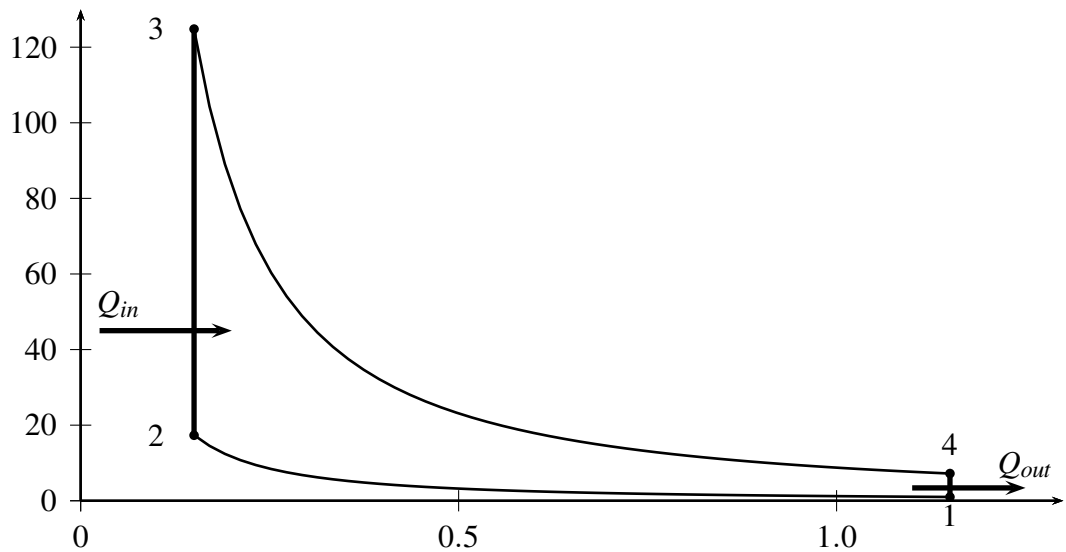


Figure 6.13: Otto cycle.

Point	P	V	T
1	1.00	1.15	300
2	17.32	0.15	689
3	124.8	0.15	4883
4	7.21	1.15	2162

Process	dU	dQ	dW	dS
1–2	88	0	-88	0
2–3	976	976	0	0.46
3–4	-632	0	632	0
4–1	-432	-432	0	-0.46

Table 6.8: End point values of PVT with Work and Energy in the Otto cycle

A formula for the thermal efficiency of the air standard Otto cycle can also be obtained by substituting the formulas used to calculate work and heat input into equation 33.35 and taking account for adiabatic processes $T_3/T_2 = T_4/T_1$.

$$\eta = 1 - \frac{W_{net}}{Q_{input}} = 1 - \frac{T_4}{T_3} = 1 - \frac{2162}{4883} = 0.557 \tag{34.45}$$

34.3 Air Standard Diesel Cycle

The Diesel engine, invented by Rudolph Diesel in 1897, is described by a thermodynamic cycle which involves four strokes (1) an adiabatic compression followed by (2) isobaric heating, (3) adiabatic expansion and (4) isochoric cooling as illustrated in figure **??**. For this example we have

For the analysis, we will assume a piston diameter of 32.00 cm and stroke of 48.00 cm to give a swept volume of 40.51 liters, a clearance volume of 05 %, or 1.929 liters, and a compression ratio of 21, which might be typical of a large marine diesel. We will also assume that fuel is injected into the engine at the end of the adiabatic compression at a ratio of 30 parts by weight of air to fuel and taken the inlet temperature to be 300 degrees K at atmospheric pressure. The heat of combustion of the diesel fuel is $h = 10,750$ calories/gram, and the specific heat of air at constant pressure is 0.171 cal/g.

As in the case of the Otto cycle, we assume air to be an ideal gas and assign the starting points $P_1 = 1$ atm, $V_1 = 40.51$ liters and $T_1 = 300$ degrees K so that we can use the ideal gas law to find that the number of moles of air that enter the engine during the intake stroke.

$$n = \frac{PV}{RT} = \frac{(1.00)(40.51)}{(0.082057)(300)} = 1.646 moles \tag{34.46}$$

Since the molecular weight of air is 28.97 grams/mole, a total mass of $1.646 \times 28.97 = 47.677$ grams of air are admitted to the cylinder. For the second point, we first note that the volume is the clearance volume and use the adiabatic rule to calculate the pressure and temperature.

$$V_2 = 1.93\ liters \tag{34.47}$$

$$P_2 = P_1 \left(\frac{V_1}{V_2}\right)^{\gamma} = (1.00)\left(\frac{40.51}{1.929}\right)^{1.4} = 70.98\ atm \tag{34.48}$$

$$T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\gamma-1} = (300)\left(\frac{40.51}{1.929}\right)^{0.4} = 1014\ K \tag{34.49}$$

For the third point, we must have the heat admitted to the cylinder through combustion of the diesel fuel when injected. Since air weighs 28.97 g/mole and we took the ratio of air to fuel injected to be 30, the total mass of the diesel fuel admitted to the cylinder is $47.677/30 = 1.589$ g. Using the heat of combustion for diesel fuel $h = 10,750$ calories/gram, we can obtain the heat released in the cylinder.

$$\Delta Q = m_{fuel} h = (0.0920)(10607) = 976\ calories \tag{34.50}$$

To calculate the temperature at point (3), we use the mass of air in the cylinder and the specific heat of air at constant volume.[10]

$$T_3 = \frac{\Delta Q}{m_{air} C_v} + T_2 = \frac{976}{(1.35)(0.171)} + 678 = 4883\ K \tag{34.51}$$

[10]For the air-standard Diesel cyle, $\Delta Q = m_{fuel} h = m_{air} C_v (T_3 - T_2)$.

Then noting that for the isobaric process, $P_2 = P_3$ and taking the ratio of the ideal gas law applied at points (3) and (2), we obtain the volume at point (3).

$$V_3 = V_2 \left(\frac{T_3}{T_2}\right) = (1.93)\left(\frac{2506}{1014}\right) = 4.77\ atm \tag{34.52}$$

The temperature and pressure at point (4) is then obtained using the adiabatic rule after noting that the $V_4 = V_1$.

$$P_4 = P_3 \left(\frac{V_3}{V_4}\right)^{\gamma} = (70.98)\left(\frac{4.77}{40.51}\right)^{1.4} = 3.55\ atm \tag{34.53}$$

$$T_4 = T_3 \left(\frac{V_3}{V_4}\right)^{\gamma-1} = (2506)\left(\frac{4.77}{40.51}\right)^{0.4} = 1065\ K \tag{34.54}$$

Now the changes in internal energy, the heat absorbed and released and the work done by and on the system can be calculated. First the change in internal energy is calculated from the definition $m_{air}C_v\Delta T$ for each process.

$$\Delta U_{12} = (47.677)(0.171)(1014-300) = 5,837\ calories \tag{34.55}$$

$$\Delta U_{23} = (47.677)(0.171)(2506-1014) = 12,204\ calories \tag{34.56}$$

$$\Delta U_{34} = (47.677)(0.171)(1065-2506) = -11,785\ calories \tag{34.57}$$

$$\Delta U_{41} = (47.677)(0.171)(300-1065) = -6,256\ calories \tag{34.58}$$

It may be noted that the changes in internal energy sum to zero for the thermodynamic process. Since the work done in the isochoric process from point (4) to (1) is zero, the heat released to the surroundings will, by the first law, equal the change in internal energy. This, however, assumes a slow conduction of heat through the cylinder walls and not the release of the gas through the exhaust valve. The work done in the adiabatic process from point (3) to (4) will, again by the first law, equal the negative of the change in internal energy. The work may, however, be calculated from

$$W_{34} = 24.2\frac{P_4V_4 - P_3V_3}{0.4} = -11,776\ calories \tag{34.59}$$

which provides a check on use of the equivalence of work and changes in the internal energy for the value. Finally, we are in a position to calculate the efficiency of the engine from the definition of efficiency, equation 33.35.

$$\eta = \frac{W_{net}}{Q_{input}} = 1 - \frac{1}{\gamma}\frac{T_3 - T_2}{T_4 - T_1} = 0.634 \tag{34.60}$$

Another formula for the thermal efficiency of the air standard diesel cycle can also be obtained by substituting the formulas used to calculate work and heat input into equation 33.35.

$$\eta = 1 - \frac{W_{net}}{Q_{input}} = 1 - \frac{1}{\gamma}\frac{T_4 - T_1}{T_3 - T_2} = 1 - \frac{1}{1.4}\frac{1065 - 300}{2506 - 1014} = 0.634 \tag{34.61}$$

For sake of completeness, the change in entropy is also calculated for each of the isochoric processes and found to be be equal but opposite in sign.

$$\Delta S_{23} = mC_v \ln\frac{T_3}{T_2} = (1.35)(0.171)ln\frac{2506}{1014} = 10.36\, calories/degree \tag{34.62}$$

$$\Delta S_{41} = mC_v \ln\frac{T_1}{T_4} = (1.35)(0.171)ln\frac{300}{1065} = -10.36\, calories/degree \tag{34.63}$$

Point	P	V	T
1	1.00	40.51	300
2	70.98	1.93	1014
3	71.0	4.77	2506
4	3.55	40.51	1065

Process	dU	dQ	dW	dS
1–2	5837	0	-5837	0
2–3	12204	17085	4878	10.36
3–4	-11785	0	11785	0
4–1	-6256	-6256	0	-10.36

Table 6.9: End point values of PVT with Work and Energy in the Diesel cycle

Problems

129. The Stirling cycle involves an isothermal process (1) to (2) at temperature 2000 ^oK, an isochoric process (2) to (3), an isothermal process (3) to (4) at temperature 300 ^oK, and an isochoric process (4) to (1). Take the high pressure to be P_1 = 20.0 atm and the low pressure to be P_3 = 1.00 atm. Work out the pressures, volumes and temperatures at the other points and calculate the changes in internal energy, the heat absorbed or lost and the work done for each process. Assume the working fluid to be 1.5 kg of Helium with a specific heat ratio $\gamma = 1.667$. Calculate and fill out tables similar to those for the Carnot and air-standard cycles in this chapter for the remaining values of P and V and the changes in internal energy, entropy, enthalpy, the work and heat flow for each process. The sums of the columns in the process chart are provided for checking answers. Answers are in kcal.

Point	P(atm)	V(liters)	T(^oK)
1	20.0		2000
2			2000
3	1.00		300
4			300

Process	ΔU	ΔQ	W	ΔS	ΔH
$1->2$					
$2->3$					
$3->4$					
$4->1$					
total	0	1390	1390	0	0

Calculate the efficiency of this Stirling engine. ans. efficiency = 39.2% without regeneration.

130. The principle difference between a Brayton cycle and the Stirling cycle is that the two isochoric processes are replaced by isobaric processes so that the four processes are an adiabatic process from points (1) to (2) an isobaric process from points (2) to (3), an adiabatic process from points (3) to (4), and an adiabatic process (4) to (1). With this information, derive an expression for the efficiency of the Brayton cycle in terms of (a) the temperatures T_1 and T_2 and (b) the pressures P_1, P_2 and the specific heat ratio. ans. (a) $\eta = 1 - \frac{T_1}{T_2}$ and (b) $\eta = 1 - \left(\frac{P_1}{P_2}\right)^{(1-\gamma)/\gamma}$

131. Draw TS-diagrams for each example in this section where not given.

35 Thermochemistry

Enthalpy as defined by equation 32.1 plays an important role in physical chemistry and chemical engineering. It is easy to conclude from its definition that a change in enthalpy at constant pressure equals the heat absorbed by the system.

$$\Delta H = \Delta(U + PV) = \Delta U + P\Delta V = \Delta Q \tag{35.1}$$

Constant pressure processes are more common in chemistry than constant volume processes since the process can be carried out in an open vessel. The heat evolved in the oxidation of a chemical element or substance to water H_2O and Carbon dioxide CO_2 is known as the **heat of combustion** and is the change in enthalpy of the reactants. For example, the oxidation reaction for graphite to Carbon dioxide is often written as

$$C + O_2 \longrightarrow CO_2 \qquad \Delta H = -94.0518 \text{ kcal} \tag{35.2}$$

where $\Delta H = -94.0518$ kcal is the heat evolved in the reaction, or the heat of combustion, which can be measured in a calorimeter. This heat of combustion

can be taken as the **heat of formation** for Carbon dioxide if the enthalpy of the elements Carbon and Oxygen are taken to be zero in their naturally occurring state. It is possible to combine two or more reactions for which the heats of combustion have been measured to obtain the heat of formation for a compound when that heat of formation is difficult to measure in the laboratory. For example, suppose that the heat of combustion of the following reactions in which Sulfur is combined with Oxygen and Water to form Sulfuric acid, H_2SO_4, are known.[11]

$$S + O_2 \longrightarrow SO_2 \qquad \Delta H = -70.96\,kcal \tag{35.3}$$

$$SO_2 + \frac{1}{2}O_2 \longrightarrow SO_3 \qquad \Delta H = -23.49\,kcal \tag{35.4}$$

$$SO_3 + H_2O \longrightarrow H_2SO_4 \qquad \Delta H = -31.14\,kcal \tag{35.5}$$

$$H_2 + \frac{1}{2}O_2 \longrightarrow H_2O \qquad \Delta H = -68.32\,kcal \tag{35.6}$$

Adding these reactions and their heats of combustion together gives the heat of formation of Sulfuric acid from its component elements.

$$H_2 + S + 2O_2 \longrightarrow H_2SO_4 \qquad \Delta H = -193.91\,kcal \tag{35.7}$$

The change in enthalpy for a given reaction can also be determined in a similar manner. Suppose that the heat of formation for Fe_2O_3 is known to be -196.5 calories and Al_2O_3 is known to be -399.1 calories. Then the change in enthalpy for a reaction in which Aluminum replaces Iron in the oxidized state in the reaction

$$Fe_2O_3 + 2Al \longrightarrow Al_2O_3 + 2Fe \tag{35.8}$$

can be determined by adding these heats together.

$$\Delta H = (-399.1 + 0) - (-196.5 + 0) = -202.6\,kcal \tag{35.9}$$

When a chemical reaction produces gases products and the system is allowed to expand at constant pressure during the reaction, the system does work against its surroundings. In this case, the change in enthalpy for the reaction must be written:

$$\Delta H = \Delta(U + PV) = \Delta U + P\Delta V = \Delta U + (\Delta n)RT \tag{35.10}$$

In this formula, ΔH is the **heat of reaction at constant pressure**, ΔU is the **heat of reaction at constant volume**, $(\Delta n)RT$ is the work done by the system and Δn

[11] Heats of combustion extrapolated from heats of formation in Handbook of Chemistry and Physics, 61st edition, 1981, CRC Press.

is the change in the number of moles of gas. Heats of reactions are normally needed at constant pressure because reactions are normally carried out at constant pressure. However, heats of combustion of liquids and solids may be measured at constant volume. In this case equation 35.10 can be used to convert data from constant pressure to constant volume.

Problems

132. Given that the heat of combustion of n-octane is 1302 kcal/mole, calculate the heat of formation of n-octane from the following reaction taking the heat of formation of O_2 to be zero, CO_2 to be 94.0518 kcal/mole and H_2O to be 68.314 kcal/mole. ans. 1367 kcal/mole

$$C_8H_{18}(g) + 12\frac{1}{2}O_2(g) \longrightarrow 8CO_2(g) + 9H_2O(\ell) \tag{35.11}$$

133. Given the heat of combustion of n-hexane, C_6H_{14}, is 995,000 cal/mole at constant pressure. What is the heat of combustion at constant volume? ans. 992,928 calories/mole

Chapter 7

ELECTROSTATICS

Electrostatics can be considered the study of non-gravitational forces between objects in the case where the agent of force is charge. The first recorded evidence of such forces is found in 600 BC, but the earliest definitive studies were made in the 17th and 18th centuries by several scientists including such notable personages as Michael Faraday and Benjamin Franklin. The agent of the electrostatic force is called the electric charge, which is known to be quantitized and conserved. Michael Faraday was the first to note the discrete nature of electric charge and Robert Millikan was first to measure the elementary charge in his oil-drop experiment.

In this study, the fundamental building blocks of matter are taken to be the electron, proton and neutron. The fundamental unit of charge is considered to be the **coulomb** with the electron possessing 1.602×10^{-19} coulombs of negative charge and the proton possessing 1.602×10^{-19} coulombs of positive charge. The neutron possesses no charge. The charge carried by electrons and protons was thought to be the smallest unit of charge until the second half of the 20th century when it was realized that the mass unit from which neutrons and protons are built, called quarks, have units of charge equal to $\frac{1}{3}$ that of the electron and proton. Particles possessing quantities of charge of the same sign repel one another while particles possessing quantities of charge of opposite sign attract one another.

36 Electric Force

The first publication describing the forces between charged particles was due to the French physicist **Charles Augustin de Coulomb** (1736-1806), who published

his work in 1785. Coulomb's law may be stated as *The magnitude of the electrostatic force between two point electric charges is directly proportional to the product of the magnitudes of each of the charges and inversely proportional to the square of the distance between the two charges.* This statement can be expressed mathematically by

$$F = k\frac{q_1 q_2}{r^2} \tag{36.1}$$

where r is the distance between the charges and q_i is the magnitude of each charge.

$$k = 8.987551787 \times 10^9\, N - m^2/C^2. \tag{36.2}$$

According to Coulomb's Law, two point charges of +1 C, one meter apart, would experience a repulsive force of 9×10^9 N, a force approximately equal to 20 million tons! As an example of this force, the electric force may be compared to the gravitational force between an electron and proton separated by a distance of 1 cm. The electron and proton each have equal charges of 1.602×10^{-19} coulombs while the mass of the electron is 9.11×10^{-31} kg and the mass of the proton is 1.67×10^{-27} kg.

$$F_e = 8.99 \times 10^9 \frac{(1.602 \times 10^{-19})^2}{(0.01)^2} = 2.36 \times 10^{-24}\, N \tag{36.3}$$

$$F_g = 6.67 \times 10^{-11} \frac{(9.11 \times 10^{-31})(1.67 \times 10^{-27})}{(0.01)^2} = 1.01 \times 10^{-63}\, N \tag{36.4}$$

This shows the electrostatic force between the electron and proton to be 10^{39} times stronger than the gravitational force. The distance between the electron and proton has no effect on this ratio since it is the same; however, the masses of the particles can have a significant effect.

Coulomb's law is an empirical law which Coulomb determined by making measurements with a torsion balance. He could not, with the apparatus available to him in the eighteenth century, prove that the exponent in the denominator of equation 36.1 is exactly equal to the integer 2. Proof of this exactness has been the subject of numerous experiments since Coulomb introduced the law. If the exponent differs from an integer value, there would be serious consequences to our modern formulation of electromagnetism and quantum electrodynamics. Current measurements have established that the exponent is indeed equal to the integer 2 within one part in 10^{16}, good enough to validate our modern theories, but there will always be a question.

Problems

134. What is the electrostatic and gravitational forces between an electron and a proton in the hydrogen atom separated by a distance equivalent to the Bohr radius, 5.29×10^{-11} meter? ans. 8.25×10^{-8} N, 3.63×10^{-47} N

135. Calculate the velocity at which the electron must travel in a circular path around a proton at a distance equal to the Bohr radius so that the centripetal force will balance the gravitational and electrical attraction holding the electron in orbit around the proton. ans. $4.59\ x\ 10^{-14}$ m/sec gravitational and $2.19\ x\ 10^{6}$ m/sec electrical

37 Electric field

The electric force, like the gravitational force, does not depend on any physical connection between the two bodies. Therefore it is necessary to assume that one charge produces a field which acts on the other charge, very similar to the concept of a gravitational field. This field is called the **electric field** with magnitude defined by:

$$E = k\frac{q_1}{r^2} \quad Newtons/coulomb \tag{37.1}$$

where q_1 is the magnitude of the charge on the body and r is the distance from the body. The electric field $\vec{\mathbf{E}}$ is a vector quantity, and the units of the electric field are Newtons/coulomb. In the case of positive charges the lines of force radiate outward, while in the case of negative charges the lines of force radiate inward. Another electric charge q_2 in this electric field will experience a force of magnitude

$$F = q_2 E \tag{37.2}$$

parallel to the direction of the field directed towards q_1 if it is of opposite sign and away from q_1 if its charge is of the same sign. As an example of the strength of this field, we can calculate the distance between an electron and a proton where the Coulomb force of attraction equal to the gravitational force of the earth on the electron.

$$r = \sqrt{\frac{kq_p}{m_e g}} = \sqrt{\frac{(8.99 \times 10^9)(1.602 \times 10^{-19})}{(9.11 \times 10^{-31}\ kg)(9.81\ N/kg)}} = 5.08\ M \tag{37.3}$$

indicating that the earth's gravitational force has little effect on separating the charges of an atom or molecule, which are 10^{11} times closer. An evaluation of the electric field of several charge distributions will prove helpful in understanding electromagnetism.

1. The **electric field of a collection of charges** can be obtained by vector addition of the fields. Consider for example the electric field at the origin of a coordinate system surrounded by point charges of magnitude q_i at distances $\vec{r}_i$ from the origin.

$$\vec{E} = \sum k \frac{q_i}{r_i^2} \hat{r}_i \tag{37.4}$$

2. The **electric field along a line perpendicular to the axis of an electric dipole** formed by two charges of equal magnitude separated by a distance of 2a at a distance r from the axis will be a vector parallel to the axis of the dipole with magnitude.

$$E_{1y} = k \frac{q}{a^2 + r^2} \frac{a}{\sqrt{a^2 + r^2}} \tag{37.5}$$

$$E_{2y} = k \frac{q}{a^2 + r^2} \frac{a}{\sqrt{a^2 + r^2}} \tag{37.6}$$

$$E_y = E_{1y} + E_{2y} = k \frac{2aq}{(a^2 + r^2)^{3/2}} \tag{37.7}$$

$$\tag{37.8}$$

3. The **electric field along the axis of a circle** at a distance x from the center of the circle when a charge q is uniformly distributed along the circle is obtained by adding the contributions from each segment of the circle along the circumference, noting that all the vertical components cancel and that $\cos\theta = x/\sqrt{(a^2 + x^2)^2}$.

$$E = \sum k \frac{dq}{r^2} \left(\frac{x}{\sqrt{(a^2 + x^2)^2}} \right) = k \frac{qx}{(a^2 + x^2)^{3/2}} \tag{37.9}$$

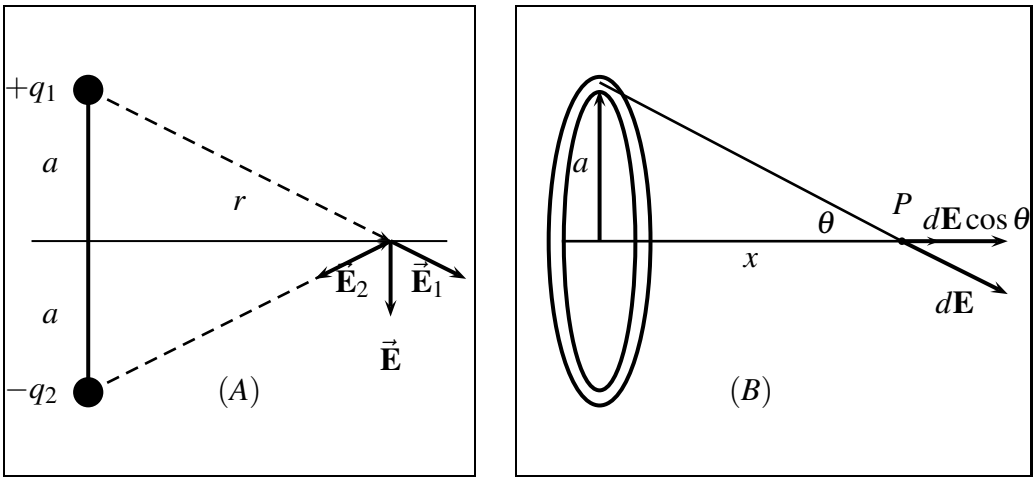


Figure 7.1: Electric field of dipole (A) and electric field of charged ring (B).

4. The **electric field of an infinitely long wire** with a charge density of λ at a distance r from the wire is inversely proportional to the first power of the distance and directed perpendicular to the wire. Noting that $dE = k\frac{\lambda dy}{z^2}$, $y = x\tan\theta$ and $z = x\sec\theta$, the resulting electric field can be calculated.

$$E = \frac{k\lambda}{x}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{2k\lambda}{x} \tag{37.10}$$

5. The **electric field from a large charged sheet** with a charged density σ at a distance r from the sheet will be a vector constant in magnitude, independent of the distance from the sheet and perpendicular to the plane of the sheet. The magnitude of the field can be calculated by integrating the field produced by a circular element of radius y and width dy over the plane of the sheet.

$$E = \int_0^\infty \frac{2\pi kx\sigma dy}{(y^2+x^2)^{3/2}} = 2\pi k\sigma \tag{37.11}$$

6. The **electric field between two conducting sheets** is constant, double the value of the field of a single sheet and zero everywhere except between the sheets. This conclusion can be reached after noting that the charge on one sheet will induce an equal and opposite charge on the other sheet and that the field is directed from one sheet to another. To summarize, the entire

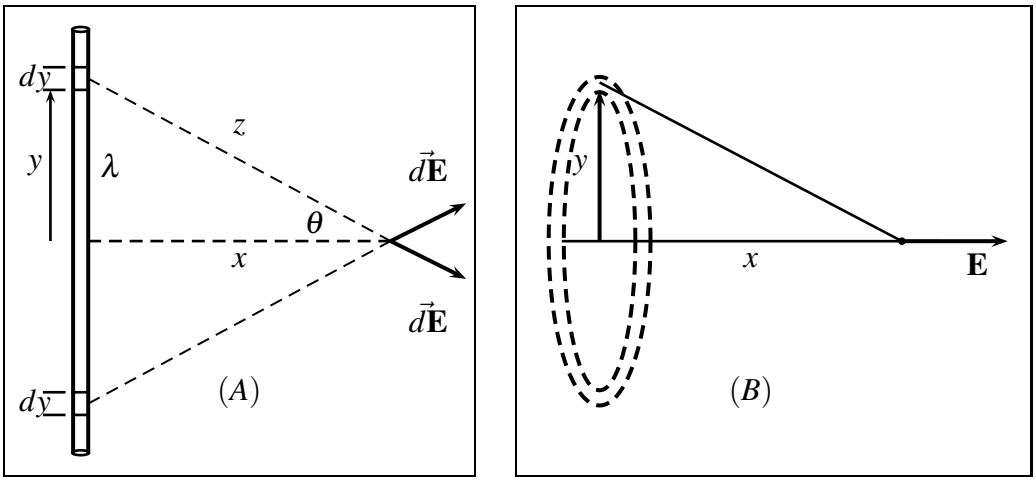


Figure 7.2: Electric field of long wire (A) and electric field of a large charged sheet (B).

electric field is confined to the space between the two oppositely charged sheets.

$$E = 4\pi k\sigma \tag{37.12}$$

Problems

136. Perform the integral to obtain the electric field along the axis of a circle.

137. Perform the integral to obtain the electric field along a line perpendicular to a long charged wire.

138. Perform the integral to obtain the electric field of a large charged sheet.

139. Suppose that an infinitely large charged sheet holds a charge of 1 microcoulomb per square meter. What is the mass of a small ball carrying 1 microcoulomb that will be suspended in the electric field above the charged sheet? ans. 5.76 g

38 Gauss' Law

Gauss' law was derived by the German mathematician and physicist **Carl Friedrich Gauss** (1777-1855) while working with the gravitational force. This law states that the flux of the electric field due to a charge distribution q over any closed surface surrounding the charge distribution can be written as

$$\varepsilon_o \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = q \tag{38.1}$$

where $\varepsilon_o = 8.85 \times 10^{-12}\ C^2/NM^2$ is the **permittivity of free space**, $\vec{\mathbf{E}}$ is the electric field and $d\vec{\mathbf{S}}$ is a unit vector perpendicular to the surface. Gauss formulated his law in 1835, but did not publish it until 1867. It is now known as one of the four Maxwell's equations which form the basis of classical electrodynamics.

Gauss' law can be used to derive the electric field of a point charge simply by integrating over the surface of a sphere of radius r at the center of which is a charge q. Noting that the electric field is constant in magnitude at equal distances from the charge and that $\vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = EdS$,

$$\varepsilon_o \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \varepsilon_o E \oint dS = 4\pi\varepsilon_o r^2 = q \quad \text{so that} \tag{38.2}$$

$$E = \frac{q}{4\pi\varepsilon_o r^2} \tag{38.3}$$

The constant $1/4\pi\varepsilon_o$ is numerically equivalent to the constant k in equation 36.2, completing the derivation, which is accepted as an indirect proof of Coulomb's law.

Gauss' law can be used to easily evaluate the electric field of each of the five charge configurations discussed in section 37 by constructing a closed surface that encompasses any specific charge distribution. In case the charge distribution is large so that the horizontal components of the electric field parallel to the charge distribution cancel, a closed surface can be constructed around a portion of the charge distribution. For example, evaluation of the electric field at the surface of a charged sheet with a surface charge density σ by constructing a pillbox on the surface of the sheet is illustrated in figure 7.3(A). The horizontal components of the electric field will cancel so that the resulting direction of the electric field vector is perpendicular to the surface. Since we have assumed a large, thin sheet we must also assume that the electric field on the opposite side of the sheet is the

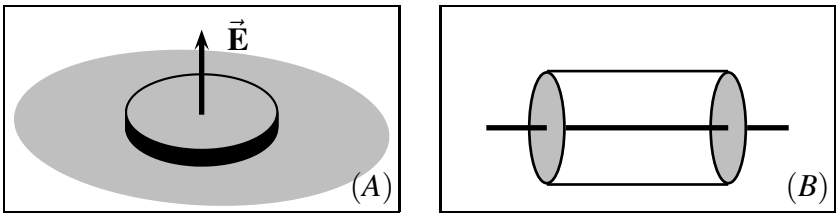


Figure 7.3: Application of Gauss' law to find the electric field at the surface of a charged conductor and around a long wire.

same. Evaluating Gauss' law over the surface results in:

$$\varepsilon_o \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = S\sigma \tag{38.4}$$

$$2\varepsilon_o ES = S\sigma \tag{38.5}$$

$$E = \frac{\sigma}{2\varepsilon_o} \tag{38.6}$$

If we had assumed a charged conductor with a closed surface in which the electric field would be zero, evaluation of the surface integral would have resulted in an electric field strength twice the value of the charged sheet.

$$E = \frac{\sigma}{\varepsilon_o} \tag{38.7}$$

As another example consider the electric field of a long charged wire, illustrated in figure 7.3(B), in which a cylinder of radius r is constructed to enclose a length L of the wire. The horizontal components of the electric field will cancel so that the electric field is perpendicular to the wire at all points and the integral $\varepsilon_o \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \varepsilon_o 2\pi rLE$ along the sides of cylinder and 0 at the ends. Gauss' law then gives the same result as in equation 37.10.

$$\varepsilon_o \pi rLE = \lambda L \quad \text{so that} \tag{38.8}$$

$$E = \frac{\lambda}{2\pi\varepsilon r} \tag{38.9}$$

Gauss' law can also be used to quickly evaluate the electric field outside and inside a uniformly charged spheroid as illustrated in figure 7.4 by replacing the

charge q with a charge integral

$$\varepsilon_o \oint \vec{\mathbf{E}} \cdot \vec{d\mathbf{S}} = 4\pi\rho \int r^2 dr \quad \text{where} \tag{38.10}$$

$$q = 4\pi\rho \int_0^a r^2 dr = \frac{4}{3}\pi\rho a^3 \tag{38.11}$$

Three charge configurations are illustrated in figure 7.4. The first consists of a charged spherical conductor, the second of a charged spherical shell and the third of a uniformly charged non-conducting spheroid. If, in figure 7.4(A), the charge integral is evaluated at a radius r greater than the radius of the shell a, the contribution of spherical volume elements with $r > a$ is zero and the charge integral becomes the total charge q. The non-radial components of the electric field cancel one another; therefore the electric field is the same as if the charge distribution were concentrated at the centroid. All the charge will reside on the surface of the conductor so that the Gaussian surface evaluated inside the sphere for $r < R$ will enclose no charge with the result that the field inside the charged conducting spheroid is zero.

$$E = \frac{kq}{r^2} \quad r > R \tag{38.12a}$$

$$E = 0 \quad r < R \tag{38.12b}$$

If, in figure 7.4(B), the charge integral is evaluated at a radius r inside the hollow shell, the result will be zero since none of the charge is located within the volume of integration and the electric field is found to be zero. Outside the conducting field, the result will be the same as if all the charge is concentrated at the centroid. In this case the results are identical to those for a charged conducting spheroid.

$$E = \frac{kq}{r^2} \quad r > R \tag{38.13a}$$

$$E = 0 \quad r < R \tag{38.13b}$$

If, in figure 7.4(C), the charge integral is evaluated at a radius $r > R$, all the charge will be contained inside the Gaussian surface and the electric field will be the same as for the other two cases. If the charge integral is evaluated at a radius $r < R$ the result will contain only the charge located within the volume defined by

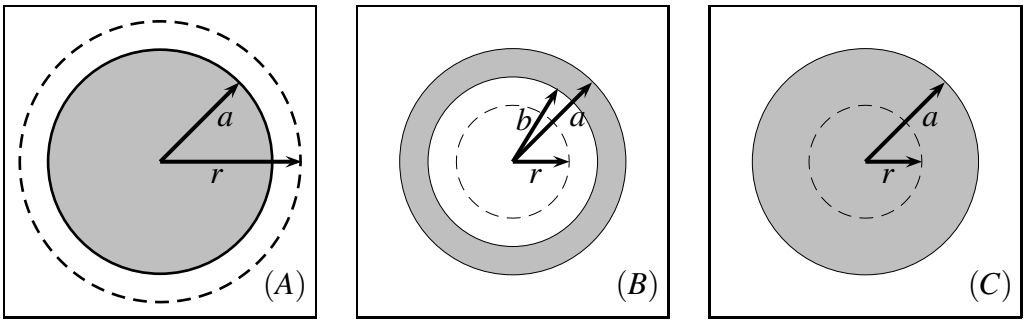


Figure 7.4: Electric field outside a uniformly charged conducting spheroid (A), inside a hollow charged shell (B) and inside a uniformly charged non-conducting spheroid (C).

r, $q' = q(r/a)^3$ and the electric field will be reduced but not zero.

$$E = \frac{kq}{r^2} \quad r > R \tag{38.14a}$$

$$E = \frac{kq}{r^2}\left(\frac{r}{a}\right)^3 = \frac{kq}{a^3}r \quad r < R \tag{38.14b}$$

$$. \tag{38.14c}$$

This results are identical to the gravitational fields inside a uniformly dense spheroid and inside a hollow shell. Therefore, in analogy to the gravitational field inside a uniformly dense spheroid, it may be easily concluded that the electric field at a point r inside a uniformly distributed charge distribution will depend only on the amount of charge inside the radius r.

It should be mentioned at this point that Gauss' theorem can be used to derive the electric field of a collection of point charges located randomly around a center of charge provided that the field is derived at a distance from the center of charge that is large in comparison to the separation of any one charge from the center of charge. We will return to this point when we undertake the study of electric potential energy in the next section.

Problems

140. An electron is placed at the center of a large conductor inside a cavity of radius 10 cm. What is the electric field 5 cm and 15 cm from the charge? ans. 5.76×10^{-7} N/C, 0 N/C

141. If the body was nonconducting and uniformly charged with a charge density $\rho = 1 \times 10^{-3} C/m^3$ what would be electric field be at the 15 cm point? ans. 3.97×10^6 N/C

142. In his argument to explain alpha particle scattering by atomic nuclei, Ernest Rutherford took the charge distribution inside the atom to consist of a positive charge Ze located at a central point surrounded by a cloud of negative charge -Ze uniformly distributed over a volume of radius R. Using this model obtain by direct calculation and Gauss' law an expression for the electric field a distance r from the central point. ans. $E = \frac{Ze}{4\pi\varepsilon_o}\left(\frac{1}{r^2} - \frac{r}{R^3}\right)$

39 Electric potential

It is convenient to introduce the electric potential from the theoretical standpoint first while drawing an analogy to gravitational potential energy.

In the chapter on linear dynamics, gravitational force between two objects M and m was defined as proportional to the product of the masses of two objects and inversely proportional to the square of the distance between their centers of gravity, $F = GMm/r^2$, with the gravitational field of M, $g = GM/r^2$, defined so that the gravitational force acting on m could be written as $F = mg$. When the line integral of $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ was also found to be independent of the path and a function of only the end points, we defined the gravitational force to be the negative gradient of a scalar function of position $U(r)$ so that $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = U(b) - U(a)$. In this case we defined the force as conservative and determined that a necessary and sufficient condition for $\vec{\mathbf{F}}$ to be conservative was that the curl of F vanish, $\nabla \times \vec{\mathbf{F}} = 0$.

We are faced with an identical situation in electrostatics. We have defined the Coulomb force between two charges Q and q as proportional to the product of the charges and inversely proportional to the square of the distance between their centers of charge, $F = kQq/r^2$. We have also defined an electric field $E = (kQ/r^2)$ so that the electric force acting on q can be written as $F = qE$. We also derived the electric field strength for several charge distributions and it is easy to see that the electric field strength can be written as the negative gradient of a scalar function of position qV in each of these cases. We can therefore expand our terminology in electrostatics to obtain comparable terminology with gravitation.

We can quickly gain working familiarity with these terms by examining each

Parameter	Gravitational	Units	Electrostatics	Units
Force	$F = -\frac{GMm}{r^2}$	Newtons	$F = -\frac{kQq}{r^2}$	Newtons
Field Strength	$g = \frac{GM}{r^2}$	Newtons/kg	$E = \frac{kQ}{r^2}$	Newtons/Coulomb
Line integral	$\int_1^2 \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = U(b) - U(a)$	Joules	$\int_1^2 \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = U(b) - U(a)$	Joules
Potential energy	$U = \frac{GMm}{r} = mV$	Joules	$U = \frac{kQq}{r} = qV$	Joules
Gradient	$F = \nabla U$	Newtons	$F = \nabla U$	Newtons
Potential function	$V = \frac{GM}{r}$	Joules/kg	$V = \frac{kQ}{r}$	Joules/Coulomb
Gradient	$g = -\frac{dV}{dr}$	Newtons/kg	$E = -\frac{dV}{dr}$	Newtons/Coulomb
Curl of Force field	$\nabla \times \vec{\mathbf{F}} = 0$		$\nabla \times \vec{\mathbf{F}} = 0$	

Table 7.1: Comparison of gravitational and electrostatic terms.

for a point charge Q. Beginning with the potential energy term,

$$U = \frac{kQq}{r} \quad [Joules]. \tag{39.1}$$

It is apparent that the Coulomb force is the negative gradient of the potential energy function

$$F = \nabla U = -\nabla \frac{kQq}{r} = \frac{kQq}{r^2} \tag{39.2}$$

and that the work done by the electric force in moving the charges closer together depends on the value of the electric potential energy at the end points

$$\int_1^2 \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_1^2 \frac{kQq}{r^2} dr = \left[-\frac{kQq}{r} \right]_1^2 = U_e(1) - U_e(2) \tag{39.3}$$

We can go one step further and write an equation for the electric potential

$$V = \frac{kQ}{r} \quad \left[\frac{Joules}{Coulomb} \right] \tag{39.4}$$

so that the electric potential energy is obtained from the electric potential simply by multiplying by the charge q

$$U = qV = \frac{kQq}{r} \tag{39.5}$$

and the electric field is the negative gradient of the electric potential

$$E = -\frac{dV}{dr}. \tag{39.6}$$

The potential at a distance r from a charge Q can also be evaluated by performing the line integral of the electric field between infinity and that point

$$V(r) = -\int_{\infty}^{r} \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = -kQ \int_{\infty}^{r} \frac{dr}{r^2} = \left[\frac{kQ}{r}\right]_{\infty}^{r} = \frac{kq}{r}. \tag{39.7}$$

Finally, we note that the necessary and sufficient condition for the electric force field to be conservative is satisfied

$$\nabla \times \vec{\mathbf{F}} = kQq\nabla \times \frac{\hat{\mathbf{r}}}{r^2} = 0 \tag{39.8}$$

39.1 Potential of a Collection of Charges

These equations work for all the fields discussed in the preceding sections. The scalar electric potential makes it easier to calculate the electric fields of irregular charge distributions which would be difficult to calculate using Gauss' law and cumbersome using vector field quantities. We simply use the scalar definition of electric potential to derive an expression for the potential due to each charge and differentiate to obtain the electric field and force. For example, consider charges of $2C$, $6C$, *and* $4C$ located at coordinates $(1,1)$, $(4,3)$ and $(2,4)$ as illustrated in figure 7.5

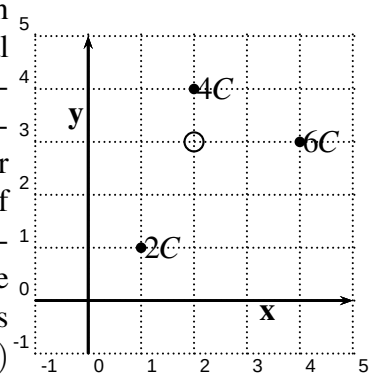


Figure 7.5: Calculating the potential of a group of charges.

Now suppose we want the potential at an arbitrary point (x,y). The summation of the potential at this point due to each charge yields

$$V = \frac{2k}{\sqrt{((x-1)^2+(y-1)^2)}} + \frac{6k}{\sqrt{((x-4)^2+(y-3)^2)}} + \frac{4k}{\sqrt{((x-2)^2+(y-4)^2)}} \tag{39.9}$$

On defining the arbitrary point (x,y) as (2,3) as indicated by the circle in 7.5, substitution into equation 39.9 will yield

$$V = 4.02 + 4.5 + 9.0 = 17.52\, Volts. \tag{39.10}$$

The components of the electric field can also be calculated as follows from equation 39.9

$$E_x = -\frac{\partial V}{\partial x} = k\left[\frac{2(x-1)}{\sqrt{((x-1)^2+(y-1)^2)^3}} + \frac{6(x-4)}{\sqrt{((x-4)^2+(y-3)^2)^3}} + \frac{4(x-2)}{\sqrt{((x-2)^2+(y-4)^2)^3}}\right] \tag{39.11}$$

$$E_y = -\frac{\partial V}{\partial y} = k\left[\frac{2(y-1)}{\sqrt{((x-1)^2+(y-1)^2)^3}} + \frac{6(y-3)}{\sqrt{((x-4)^2+(y-3)^2)^3}} + \frac{4(y-4)}{\sqrt{((x-2)^2+(y-4)^2)^3}}\right] \tag{39.12}$$

This approach to derivation of the electric potential can be extended to cover such charge configurations as electric dipoles, quadrapoles, octapoles, etc which have played such an important role in the development of physics. Derivation of these electric potentials is made the subject of problems at the end of this chapter.

39.2 Potential of a Continuous Charge Distribution

The line integral can be performed for a long charged wire, except in this case the zero point must be taken at some point b instead of infinity to evaluate the integral.

$$V(r) = -\int_b^r \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = -2k\lambda \int_b^r \frac{dr}{r} = [2k\lambda \ln r]_b^r = -2k\lambda \ln \frac{b}{r} \tag{39.13}$$

In the case of a continuous charge distribution such as a uniformly charged disk, the potential at a distance r from the disk can be calculated by dividing the disk into charge elements and integrating over the surface of the disk.

Letting the charge inside the ring of radius y and width dy as illustrated in figure 7.6 be represented by $dq = 2\pi y\sigma dy$ so that the potential at a point r along a perpendicular axis to the disk is

$$dV = k\frac{2\pi\sigma y dy}{\sqrt{r^2+y^2}}. \tag{39.14}$$

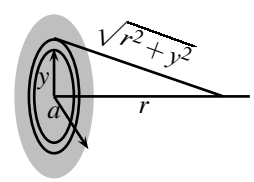


Figure 7.6: Calculating the potential of a charged disk.

The potential of the disk at r can be obtained by integration over the surface from y=0 to y=a to obtain

$$V = \frac{\sigma}{2\varepsilon}\left[\sqrt{r^2+a^2} - r\right] \tag{39.15}$$

$$= \frac{q}{4\pi\varepsilon r}, \tag{39.16}$$

after the binomial theorem is used to reduce the quantity in brackets and the total charge on the disk is defined to be $q = \pi a^2\sigma$.

Further understanding of the electric potential of a charge distribution can be gained by an examination of the electric potential V(r) and electric field strength E(r) for each of the charge distributions examined in the previous section. For this purpose, figure 7.4 has been redrawn as figure 7.7 to illustrate the variation of both

V(r) and E(r). In this figure, the radius of the charge distribution is indicated by a dotted line at $r = a$. The scale of the vertical axis is arbitrary since there would be a large difference in the magnitudes of V(r) and E(r). The electric field in each diagram is the same as in figure 7.4.

In figure 7.7(A), all the charge has migrated to the surface of the spheroid under the influence of the electric field. Outside the spheroid, the potential V(r) is the same as if the total charge were concentrated at the centroid. Inside, the potential is constant and equal the potential at the surface because the potential must be continuous at boundaries. This is why the state of charge of a conductor is usually described by stating its potential.

In figure 7.7(B), the charge distribution is assumed to be restricted to the surface of the sphere. The potential V(r) is the same as for the charged conducting spheroid.

In figure 7.7(C), the spheroid is assumed to be non-conducting and that the charge is distributed with uniform density throughout the spheroid. Outside the charged non-conducting spheroid, the potential is the same as if the total charge were concentrated at the centroid of the distribution. However, the potential inside the spheroid is not so easily evaluated. Inside the spheroid, the electric field is known and can be used to calculate the potential with the line integral to obtain:

$$E(r) = kQ\frac{r}{R^3} \tag{39.17a}$$

$$V(r) = -\int_r^R \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = \frac{kQ}{2R}\left[3 - \frac{r^2}{R^2}\right], \tag{39.17b}$$

At the surface the equations for the potential inside and outside reduce to the same value.

We can use this information to deduce the surface charge density on an irregularly shaped conductor. We know that if a conductor is charged to a high voltage arcing will first occur at the sharpest points of the conductor. This is called **corona discharge**. We also know that arcing results from ionization of the air around the conductor and that ionization of the air is brought about by an intense electric field. Since the electric field is the gradient of the potential, we therefore know that the potential gradient is higher at sharp points; and since the magnitude of the electric field is proportional to the surface charge density, we therefore expect that surface charge density will be higher near the sharp points of a conductor. This reasoning flows from a knowledge base and logical reasoning. To conclude that the surface charge density is greater near sharp points and reverse the logic to predict corona

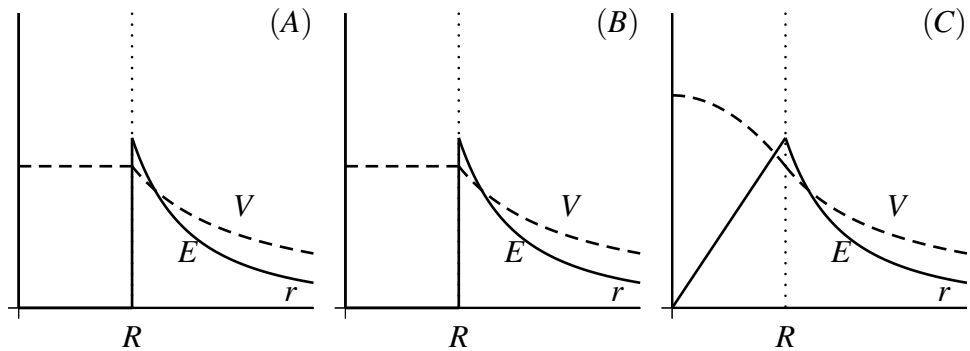


Figure 7.7: Electric field and electric potential outside a charged conducting spheroid (A), inside an outside a hollow charged shell (B) and inside and outside a uniformly charged non-conducting spheroid.

discharge is more difficult. We can, however, argue that the electric potential outside a conductor tends to equalize with the potential at the surface thus resulting in a higher potential gradient and work backwards to the higher electric field and corona discharge as well as higher surface charge densities at sharp points. This is often the logic taken in classroom discussions.

39.3 Potential Difference

The electric potential is so useful in electrical work that a special unit has been defined for it.

$$1 \text{ Volt} = 1 \text{ Joule/Coulomb} \tag{39.18}$$

We may therefore say that energy in the amount of 1 Joule/Coulomb is expended in moving a charge from one point to another that is higher in potential by 1 volt. Often, we work with the difference in potential between two points and note that the difference in potential energy of a charge in the potential field between those two points is the product of that charge and the potential difference between the two points.

$$\Delta U = q\,(V_2 - V_1) \tag{39.19}$$

39.4 Maxwell's First Equation

Starting with Gauss' theorem, we can go further to define three additional equations that played an important role in the development of the theory of electricity

and magnetism. Writing Gauss' theorem in the following form

$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{q}{\varepsilon_o}, \tag{39.20}$$

which is the first of **Maxwell's equations**. Using the divergence theorem to express the surface integral as the divergence of the vector $\vec{\mathbf{S}}$ gives us

$$\nabla \cdot \vec{\mathbf{E}} = \frac{q}{\varepsilon_o}. \tag{39.21}$$

Then expressing $\vec{\mathbf{E}}$ as the gradient of the electric potential gives us **Poisson's equation**,

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{q}{\varepsilon_o}. \tag{39.22}$$

In the event that the charge is zero, we have **Laplace's equation**, which is central to numerous mathematical theories,

$$\nabla^2 V = 0. \tag{39.23}$$

Problems

143. Obtain by direct calculation an expression for the electric field at the mouth of a small hole of radius b in a uniformly charged spherical shell of radius R at the center of the mouth of the hole.

144. Suppose the excess outward atmospheric pressure inside an uncharged soap bubble reaches equilibrium with the inward force of surface tension at a radius 3.00 cm. (a) Take the surface tension as 30.0 dynes/cm and calculate the excess atmospheric pressure inside the soap bubble. ans. 4.00 Pa. (b) Then assume that a charge of 60.0 nanocoulombs is placed on the soap bubble. What will be the effect on the radius of the bubble?

145. Three charges are located as follows: $1\mu C$ at (2,3), $3\mu C$ at (4,4) and $4\mu C$ at (6,2), where the coordinates are in meters. What is the total potential energy of the $3\mu C$ and $4\mu C$ charges in the field of the $1\mu C$ charge? ans. 2.08×10^{-2} J

146. Given a dipole with positive charge (+q) located along the z-axis a distance a from the center point and a negative charge (-q) located a distance a along

the z-axis in the opposite direction from the center point. Calculate the electric potential a distance r from the center point with the vector r making an angle θ with the z-axis. Take the dipole moment p = 2a. ans. $k\frac{p\cos\theta}{r^2}$.

147. Calculate the magnitude of the electric field and potential a distance equal to the Bohr radius from a proton. ans. 5.15 x 10^{11} N/C, 27.2 Volts

148. What energy must be expended in moving an electron from infinity to a distance equal to the Bohr radius from a proton? ans. 27.2 eV When located in a stable orbit around the proton, what will be the potential and kinetic energies of the electron and the ionization energy? -27.2 eV, 13.6 eV and 13.6 eV

149. Given two coaxial conducting cylinders with the inner cylinder with radius a at a potential V_o and the outer cylinder with radius b grounded. Calculate the potential in the space between the cylinders a distance r from the center. ans. $V_o \ln\frac{r}{b}/\ln\frac{a}{b}$

150. Calculate the potential difference between to concentric spheres. The outer sphere of radius b has a positive charge of magnitude Q and the inner sphere of radius a has an equal and opposite induced charge. ans. $kQ\left(\frac{1}{a}-\frac{1}{b}\right)$

40 Capacitance

Capacitance is a term used to define the capacity of a given physical arrangement of conductors between which there is an electric potential difference V to hold a charge. Consider first the parallel plate arrangement of figure 7.8 in which two parallel plates each of area A are separated by a distance d. One plate carries a charge $+Q$ and the other $-Q$.

40.1 Definition

Using Gauss' law, we can obtain the electric field strength between the plates.

$$\varepsilon_o \oint_A \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = Q \tag{40.1}$$

$$\varepsilon_o EA = Q \tag{40.2}$$

$$E = \frac{Q}{A\varepsilon_o} \tag{40.3}$$

The electric field is confined to the space between the plates and is constant in value. The electric potential difference is obtained by integrating the line integral $\vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}$ between the plates.

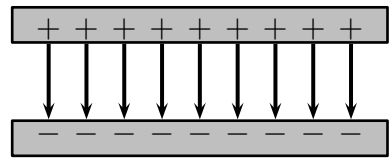


Figure 7.8: Parallel plate capacitor.

$$\int_0^d \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = V \tag{40.4}$$

$$V = Ed \tag{40.5}$$

Combining these equations gives

$$Q = CV \quad \text{where} \tag{40.6}$$

$$C = \frac{\varepsilon_o A}{d} \tag{40.7}$$

We define the constant of proportionality between Q and V, C, as the **capacitance** of the system. The unit of capacitance is the farad, named in honor of **Michael Faraday** (1791-1867) who developed the concept of capacitance. By definition, *a capacitor has a capacitance of 1 farad if the addition of 1 Coulomb increases the potential difference by 1 Volt.*[1]

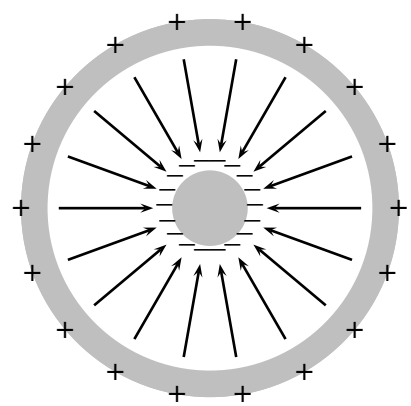


Figure 7.9: Cylindrical capacitor.

40.2 Cylindrical Capacitor

If the capacitor were cylindrical of length ℓ with the inner conductor of radius a negatively charged and the outer conductor of radius b positively charged, we could construct the Gaussian surface as a cylinder around the inner conductor at a distance r from the center extending the length of the cylinder and perform the integral to get

[1] The farad is an exceptionally large capacitance so that capacitances normally encountered are expressed in $\mu f = 10^{-6} f$ or $\mu\mu f = 10^{-12} f$.

$$\varepsilon_o \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = Q \tag{40.8}$$

$$\varepsilon_o E 2\pi r \ell = Q \tag{40.9}$$

$$E = \frac{Q}{2\pi\varepsilon_o \ell r} \tag{40.10}$$

The electric field is confined to the space between the inner and outer conductors and its magnitude decreases with distance from the inner conductor. The electric potential difference is obtained by integrating the line integral $\vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}$ between the conductors.

$$\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = V \tag{40.11}$$

$$V = \frac{Q}{2\pi\varepsilon_o \ell} \ln \frac{b}{a} \tag{40.12}$$

Combining these equations gives

$$C = \frac{2\pi\varepsilon_o \ell}{\ln(b/a)} \tag{40.13}$$

40.3 Spherical Capacitor

If the capacitor is a spherical capacitor, a similar calculation to that performed for a cylindrical capacitor will result in the following.

$$E = \frac{Q}{4\pi\varepsilon_o r^2} \tag{40.14}$$

$$V = \frac{Q}{4\pi\varepsilon_o} \frac{b-a}{ab} \tag{40.15}$$

$$C = \frac{4\pi\varepsilon_o ab}{b-a} \tag{40.16}$$

The spherical capacitor is of limited practicality but the result of the above calculation is often used to define the capacitance of an isolated sphere. For this purpose, it is assumed that the outer conductor is an infinite distance away from the inner conductor. Allowing the quantity b to approach infinity in equation 40.16 results in a definition of the capacitance of an isolated sphere.

$$C = 4\pi\varepsilon_o a \tag{40.17}$$

40.4 Energy Storage

Energy is stored in the electric field inside a capacitor. The magnitude of this energy storage can be calculated by calculating the work necessary to move charge from one plate to the other.

$$W = \int_0^Q \frac{Q'}{C} dQ' = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \tag{40.18}$$

Since this energy must be stored in the electric field, we can calculate the energy density by dividing the total energy by the volume of the capacitor. For the parallel plate capacitor with plate area A and separation of the plates d, and using equation 40.7 we obtain

$$u = \frac{U}{Ad} = \frac{1}{2}\varepsilon_o E^2 \tag{40.19}$$

A similar calculation can be performed for cylindrical and spherical capacitors.

40.5 Capacitors in Series and Parallel

When more than one capacitor is place in a series and a voltage V is applied across the string, the charge Q on each plate of each capacitor must be the same. The voltages across each capacitor will depend on the capacitance of each capacitor, but the voltages will add to be the total voltage across the string.

$$V = V_1 + V_2 \tag{40.20}$$

$$= \frac{Q}{C_1} + \frac{Q}{C_2} \tag{40.21}$$

$$= Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \quad \text{so that} \tag{40.22}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \tag{40.23}$$

If capacitors are connected in parallel and a voltage V applied across the parallel combination, the charge on each capacitor will depend on the capacitance of each capacitor, but the charges will add to be the total charge across the combination.

$$Q = Q_1 + Q_2 \tag{40.24}$$

$$= C_1 V + C_2 V \tag{40.25}$$

$$= (C_1 + C_2)V \quad \text{so that} \tag{40.26}$$

$$C = C_1 + C_2 \tag{40.27}$$

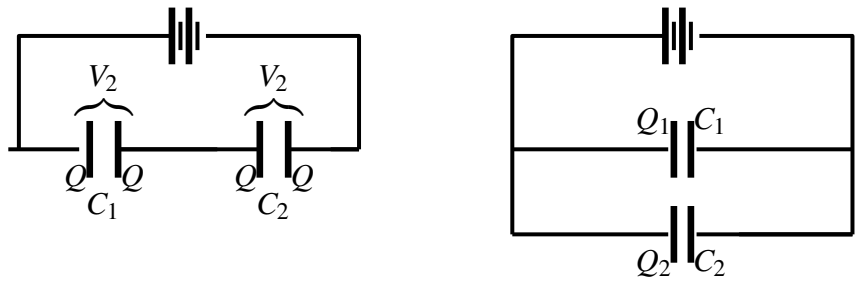


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Figure 7.10: Capacitors in series and parallel.

Problems

151. Show that the capacitance of two long oppositely charged parallel wires each of radius a and a distance d apart is $C = \pi\varepsilon_o / \ln\left(\frac{d-a}{a}\right)$

152. Show that the force between the two plates of a parallel plate capacitor is $F = V^2 \frac{\varepsilon_o A}{2d^2}$ where σ is the charge density on each plate.

153. What is the capacitance of 10 meters of a coaxial transmission line of inner radius 1 cm and outer radius 2 cm filled with air? ans. 0.653 picofarads

154. A two-wire transmission line has its wires spread 6.00 meters apart and each wire has a radius of 5.00 mm. If the potential difference between the wires is 100,000 volts, the capacitance per meter of the line, the charge on each line per meter contained in the electric field per meter? ans. 0.392 $\mu C/m$, 3.92 pF/m, 0.0196 Joules/meter.

41 Dielectrics

Michael Faraday investigated the effect of filling the space between the plates of a capacitor with various materials in 1837 and found that for the same potential difference he could place a larger charge on the capacitor. This has the effect of multiplying the capacitance by a constant factor which became known as the dielectric constant κ. For example, the capacitance of a parallel plate capacitor

with a dielectric material between the plates can be written as

$$C = \frac{\kappa \varepsilon_o A}{d} \tag{41.1}$$

This effect can be explained by recognizing that the molecules of any material will acquire, to a greater or lesser extent, a dipole configuration when placed in an electric field. Atoms are composed of nuclei concentrated in a small volume, typically less than 10^{-14} meters in radius while the electron cloud surrounding the nucleus may have a mean radius approximately 1000 times greater. When an electric field is applied, the electrons tend to concentrate at one end of the molecule while the positively charged nucleus is on the other. The molecules will become polarized in opposition to the electric field with the positive ends of the molecules being concentrated near the plate with negative charge and the negative ends of the molecules being concentrated near the plate with the positive charge. The net effect is to reduce the electric field inside the capacitor and thus the potential requiring a greater charge to be placed on the plate to maintain the same potential across the plates.

Although the dielectric constant was defined for the parallel plate capacitor, it applies for a dielectric used in any capacitor. This allows us to write in general,

$$C = \kappa C_o \tag{41.2}$$

$$E = \frac{E_o}{\kappa} \tag{41.3}$$

$$V = \frac{V_o}{\kappa} \quad \text{and} \tag{41.4}$$

$$\varepsilon = \kappa \varepsilon_o \tag{41.5}$$

where the subscripted quantities, C_o, etc, represent values in a vacumn and the unsubscripted quantities represent values in a dielectric medium where κ is the dielectric constant.

Dielectric materials play an important role in the construction of capacitors and especially in the miniaturization of electronic circuits. In general, the dielectric constant of air and most all gases is nearly 1 because of their low density and lack of rigidity. The dielectric constants of most plastics ranges from 2 to 6. That of water is about 80 because the molecules are easily polarized. Several ceramics have dielectric constants above 10 and some as high as several hundred. The dielectric constants of several common substances is listed in table 7.2.

The reduction of the electric field in a dielectric material can be described in mathematical terms by introducing the **electric polarization vector $\vec{\mathbf{P}}$** for isotropic

air	1.000590
water	80
mica	7-8
glasses	4-6
ceramics	5-5000
plastics	2-3

Table 7.2: Range of dielectric constants for common materials

dielectrics defined by

$$\vec{\mathbf{P}} = \chi_e \varepsilon_o \vec{\mathbf{E}} \tag{41.6}$$

where the dimensionless constant, χ_e is the **electric susceptibility** of the material. For isotropic dielectrics, the polarization vector and electric field are parallel. For anisotropic materials, $\vec{\mathbf{P}}$ and $\vec{\mathbf{E}}$ are not normally parallel and χ_e is not a scalar constant. The reduction of the electric field can also be described in terms of the charge σ_p which migrates to the surface of the dielectric partially neutralizing the effect of the charge on the capacitor plates.

An alternative approach to describing the effect of the dielectric would be to define the **electric displacement vector** by

$$\vec{\mathbf{D}} = \varepsilon_o \vec{\mathbf{E}} + \vec{\mathbf{P}} \tag{41.7}$$

$$= \varepsilon_o (1 + \chi_e) \vec{\mathbf{E}} \tag{41.8}$$

which leads to representation of the permittivity of the dielectric medium, ε and the dielectric constant by

$$\varepsilon = \varepsilon_o (1 + \chi_e) \tag{41.9}$$

$$k_e = \frac{\varepsilon}{\varepsilon_o} = 1 + \chi_e \tag{41.10}$$

The electric displacement can be written as proportional to the electric field strength in the dielectric medium

$$\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}} \tag{41.11}$$

which allows us to rewrite Gauss' theorem in a dielectric medium as

$$\int_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{S}} = \int_{vol} \rho d(vol) \tag{41.12}$$

and use the divergence theorem to convert the surface integral to a volume integral to obtain the differential form of Maxwell's first equation.

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{41.13}$$

Problems

155. What is the capacitance of a parallel plate capacitor with plates separated by a distance d when a dielectric k of width b is inserted between the plates equally distance from either? ans. $C = \varepsilon_o A / [(d-b) + b/k]$

156. Two sheets of Aluminum foil 40.0 cm x 20.0 m are separated by a a sheet of mica 1.00 mm thick with k= 7.50. What is the capacitance? ans. 0.531 $\mu farads$

157. What is the capacitance of a sphere of radius 1 cm submerged in water which has a dielectric constant of 80? ans. 88.9 pf

42 Electric Current and Resistance

42.1 Definitions

If an electric field is established across a conductor a charge will flow through the conductor. This charge flow is described by defining the **electric current**, I, as the rate at which charge flows past a given point.

$$I = \frac{dq}{dt} \quad \left[\frac{Coulombs}{second}\right] \tag{42.1}$$

The SI unit of current is the **ampere**, which represents the flow of 1 Coulomb of charge per second. Current is a scalar quantity, but it is often convenient to use the **current density**, **j**, a vector that defines the direction of flow of charge across a unit surface area per unit time so that

$$I = \int_S \vec{\mathbf{j}} \cdot d\vec{\mathbf{s}}. \tag{42.2}$$

It is accepted that free electrons in a conductor carry charge through the conductor. If there are n free electrons per unit volume, each with a charge $-e$ and moving with an average velocity $\vec{\mathbf{v}}$, the current density will be

$$\vec{\mathbf{j}} = -ne\vec{\mathbf{v}}. \tag{42.3}$$

Movement of charge through a conductor is a jerky motion since electrons in the conductor often collide with other electrons loosing their forward velocity and

transferring the role of moving charge to the next electron. If we take the time between collisions to be $t = \lambda/u$, where λ is the mean free path between collisions and u is the average random velocity of the electron, the electron will acquire a forward velocity $\vec{\mathbf{v}}$ under acceleration by an electric field $\vec{\mathbf{E}}$ before collision.

$$\vec{\mathbf{v}} = \vec{\mathbf{a}}t = \left(\frac{-\vec{\mathbf{E}}}{m}\right)\left(\frac{\lambda}{u}\right) = -\vec{\mathbf{E}}\frac{e\lambda}{mu} \tag{42.4}$$

Taking the average forward velocity to be one-half this value since the moving electron will stop at the time of collision and the next electron must be accelerated by the field, we obtain for the current density

$$\vec{\mathbf{j}} = -\frac{1}{2}ne\vec{\mathbf{v}} = \frac{ne^2\lambda}{2mu}\vec{\mathbf{E}} \tag{42.5}$$

We can now define the **resistivity**ρ of the conductor by

$$\rho = \frac{2mu}{ne^2\lambda} \quad \text{so that} \tag{42.6}$$

$$j = \frac{E}{\rho} \tag{42.7}$$

Since the current I flowing through the conductor can be written as jA and the voltage across the conductor of length L can be written as $V = EL$ we can write

$$V = \frac{\rho L}{A}I. \tag{42.8}$$

This prompts us to define the **resistance**, R, to the flow of charge through a conductor of cross sectional area A and length L as

$$R = \rho\frac{L}{A}, \tag{42.9}$$

and express the current flowing in the conductor as

$$V = IR \tag{42.10}$$

which is known as **Ohm's law**, named for George Simon Ohm (1787-1854) a high school teacher in Cologne who investigated the effect of applied voltage on current flow in resistive media.

The work done in taking a charge q through a potential drop of V is $W = qV$ and the rate at which work is done is $dW/dt = Vdq/dt$. The energy required to do this work must be dissipated as heat in the conductor, which allows us to write the rate of heat loss in the conductor or electric power loss as

$$P = IV = I^2R \tag{42.11}$$

The relationship between current, current density and charge density may be studied further by noting that

$$I = \frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t} dt \tag{42.12}$$

and using this expression for the current in equation 42.2 to obtain

$$\int_V \frac{\partial \rho}{\partial t} dt = \int_S \vec{\mathbf{j}} \cdot d\vec{\mathbf{s}}. \tag{42.13}$$

After using the divergence theorem to convert the surface integral to a volume integral

$$\int_V \frac{\partial \rho}{\partial t} dt = \int_S \nabla \cdot \vec{\mathbf{j}} dV, \tag{42.14}$$

it becomes apparent that

$$\nabla \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{42.15}$$

This equation is known as the **equation of continuity** and is an expression of the conservation of charge. In the event that the current flow is steady and that there is no change in the charge density, that is $\frac{\partial \rho}{\partial t} = 0$, it is then clear that the divergence of the current density is zero.

$$\nabla \cdot \vec{\mathbf{J}} = 0 \tag{42.16}$$

42.2 Kirchhoff's Laws

In 1847 Gustav Robert Kirchhoff (1824-1887), a German physicist and professor, stated two simple laws that make solving circuit problems easy. These laws are[2]

[2] Kirchhoff stated his two laws while a student for a seminar

1. At every instant the sum of the currents flowing away from a point in a circuit is equal to the sum of the currents flowing toward the point.

$$\sum I_{in} = \sum I_{out} \tag{42.17}$$

2. At every instant the algebraic sum of the emf's in any closed loop equals the algebraic sum of the IR drops in potential in that loop.

$$Emf = \sum IR_i \tag{42.18}$$

It is important to distinguish between the electromotive force (emf) and terminal voltage. Any device in which some form of energy, such as chemical or mechanical, is converted to electrical energy is called a **source of emf**. Examples of such devices are batteries, motors and generators. The magnitude of the emf in volts of a source furnishing energy is defined as the work in Joules that the source can do in sending 1 coulomb of charge completely around the circuit and back to the starting point. Alternatively, the magnitude of the emf in volts of a source which is receiving energy is defined as the electrical energy in joule3s converted in the source into some form of energy other than heat when 1 coulomb of charge flows through the source against its own emf. Any source of emf will contain some internal resistance R_s; therefore the terminal voltage or potential difference (TPD) of any source of emf is related to the emf by

$$TPD = emf - IR_s \tag{42.19}$$

42.3 Resistors in Series and Parallel

When more than one resistor is place in series and a voltage V is applied across the string, the current I flowing through each resistor is the same so that the total resistance is the sum of the resistances of each resistor.

$$V = IR_1 + IR_2 + \cdots = IR \quad \text{where} \tag{42.20}$$

$$R = R_1 + R_2 + \cdots \tag{42.21}$$

If resistors are connected in parallel and a voltage V applied across the parallel combination, the total current flowing is the sum of the currents flowing in each resistor.

$$I = \frac{V}{R_1} + \frac{V}{R_2} + \cdots = \frac{V}{R} \quad \text{where} \tag{42.22}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots \tag{42.23}$$

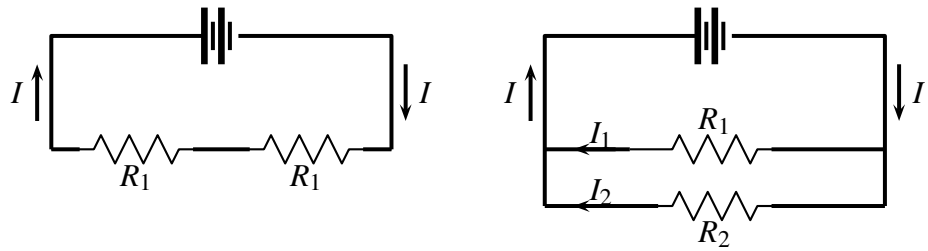


Figure 7.11: Resistors in series and parallel.

Kirchhoff's laws may be applied in several ways to calculate the current flowing through resistive elements of an electric circuit. One way, known as the method of cyclic currents, is to identify current loops in the circuit and apply Kirchhoff's second law to each loop. In this application, assignment of the direction of current flow should be consistent in all loops. One way is to always let positive current flow be out of the positive terminal of the battery and around the loop to the negative terminal. Also the net current flowing through any one resistor that happens to be common to two loops must be the algebraic sum of currents flowing through that resistor. As an example, consider the electric circuit in figure 7.12

Using this approach, two simultaneous equations would result which could be solved for the currents.

$$\xi_1 = I_1R_1 + (I_1 - I_2)R_2 + I_1R_3 \tag{42.24}$$

$$\xi_2 = I_2R_4 + (I_2 - I_1)R_2 \tag{42.25}$$

42.4 RC Networks

If a constant voltage V_o is applied to a resistor R and capacitor in series, the voltage drops across the resistor and capacitor can be described d by

$$V_o = I(t)R + V_c(t) \quad \text{where} \tag{42.26}$$

$$= I(t)R + \frac{1}{C}q(t), \tag{42.27}$$

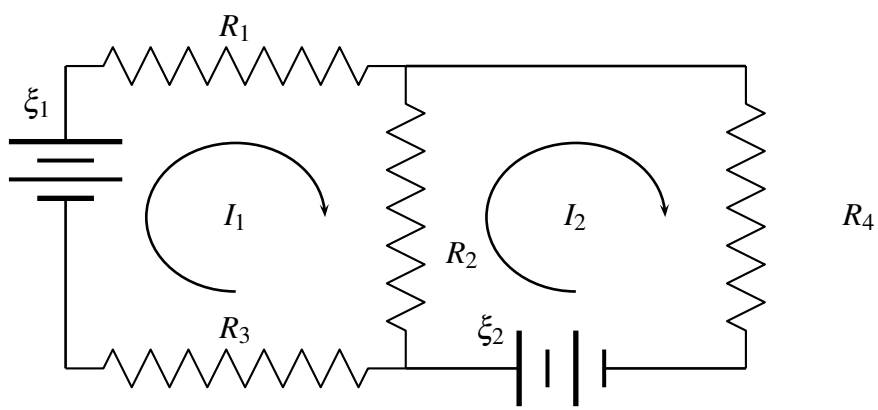


Figure 7.12: Application of Kirchhoff's second law to solving for current flow.

since $V_c = q(t)/C$. Differentiating and rearranging gives

$$RC\frac{di}{dt} + i = 0, \tag{42.28}$$

which has solution

$$i(t) = i_o e^{-t/RC}, \tag{42.29}$$

thus allowing the voltage drops across the resistor and capacitor to be written as

$$V_R = i_o R e^{-t/RC} \tag{42.30}$$

$$V_C = i_o R\left[1 - e^{-t/RC}\right]. \tag{42.31}$$

A RC network is illustrated in figure 7.13 showing the time profile of current flowing and the voltage developed across the resistor and capacitor. The initial current at $t = 0$ is $i_o = V_o/R$ before any charge is deposited on the capacitor, but the current decreases as the voltage builds up across the capacitor. Similarly, the voltage across the resistor decreases as the current decreases. While the current is flowing the voltage across the capacitor builds up until it finally equals the applied voltage V_o and the current ceases to flow.

42.5 Sources of Emf

The invention of electrochemical cells for the production of electricity are generally credited to Count Alessandro Giuseppe Antonio Anastasio Volta (1745-1827)

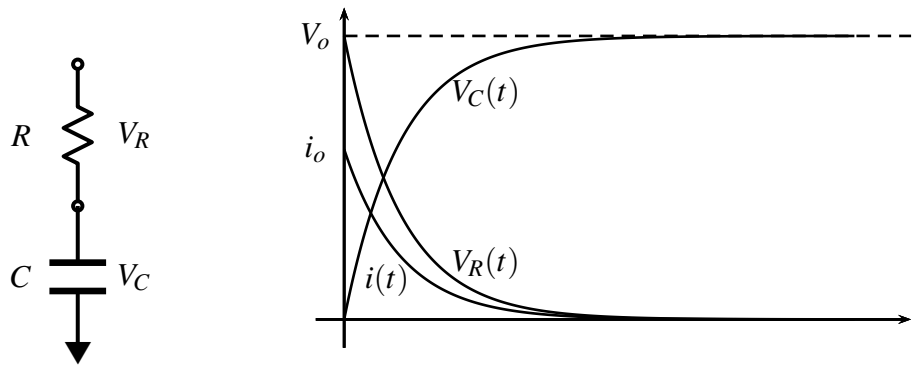


Figure 7.13: RC Network with voltage outputs taken across resistor and capacitor.

an Italian physicist. An early electrochemical cell consisted of a Zinc electrode immersed in a solution of zinc sulfate and a copper electrode immersed in a solution copper sulfate with the two solutions separated from one another by a divider as illustrated in figure 7.14.

Since each electrode is immersed in a solution of its ions, an equilibrium reaction takes place in each solution.

$$Zn \longleftrightarrow Zn^{++} + 2e^- \tag{42.32a}$$

$$Cu \longleftrightarrow Cu^{++} + 2e^- \tag{42.32b}$$

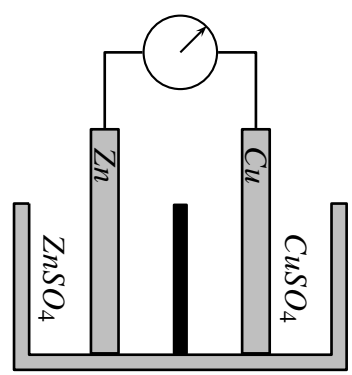


Figure 7.14: Electrochemical cell.

It turns out that Zinc has a much greater tendency to give up electrons than Copper. Therefore electrons released by the Zinc electrode as Zinc atoms go into the solution pass through the wire to the Copper electrode. At the Copper electrode the electrons are released to the surrounding copper ions which are deposited on the copper electrode. This gives rise to two important definitions.

- Since the Zinc electrode is giving up electrons, we say that it is being **oxidized** and that the reaction is a **oxidation** reaction.
- Since the Copper electrode is receiving electrons, we say that it is being **reduced** and that the reaction is a **reduction** reaction.

Our problem is to describe this process mathematically. To do so, we measure the direction of current flow or voltage developed across a resistance in the electron flow path between each reaction and a standard reference reaction which we will take to be

$$H_2 \longleftrightarrow 2H^+ + 2e^- \tag{42.33}$$

With reference to this reaction, the oxidation potential of Zinc is 0.7618 volts while that of Copper is -0.3419 volts. Using these results, we can reverse the direction of the equation for copper and write equations 42.32 as oxidation and reduction reactions and sum to get the resulting oxidation-reduction equation.

$$Zn \longleftrightarrow Zn^{++} + 2e^- \quad E_{ox} = +0.7618\ v \tag{42.34a}$$

$$\underline{Cu^{++} + 2e^- \longleftrightarrow Cu} \quad E_{red} = +0.3419\ v \tag{42.34b}$$

$$Zn + Cu^{++} \longleftrightarrow Zn^{++} + Cu \quad E_{rxn} = +1.1037\ v \tag{42.34c}$$

The difference between the ionization potentials for the reaction is $+1.100$ volts, which is the potential that is measured between the electrodes in figure 7.14. The mechanics of operation are that for each gram equivalent of Zinc that goes into solution at the Zinc electrode, 6.023×10^{23} electrons pass through the wire to the Copper electrode and one gram equivalent of Copper is deposited on the Copper electrode. Since electrons flow from the Zinc electrode it is given the name of **anode**, while the Copper electrode is given the name of **cathode**. Copper is thus deposited on the cathode while the Zinc electrode is depleted of Zinc.

For sake of clarity, the reduction of Zinc takes place at the anode and the oxidation of Copper takes place at the cathode in the reaction described above. When written as in equation 42.34a(a), the reaction is referred to as oxidation and the potential is referred to as oxidation potential. When the oxidation reaction and the sign of the oxidation potential is reversed as in equation 42.34a(b), the reaction is referred to as reduction and the potential is referred to as reduction potential. Then the sum of the potentials is referred to as reduction-oxidation, or simply as **Redox**. If the redox potential is positive the reaction can proceed simultaneously; if the redox potential is negative the reaction cannot proceed without current being forced through the solution.

Numerous types of electrochemical cells have been developed and generally fall into two categories. Electrochemical cells are called **reversible** if passing a current from a generator in the opposite direction will bring the cell back to its original state after discharge. The cells are called **irreversible** if part of the solution is no longer available for conversion such as, for example, a gas that is

evolved or an oxide that is not soluble. One of the more common reversible cells is the Lead-Acid storage battery, which is used in automobiles and can be recharged. This battery consists of a Lead anode and Lead oxide cathode both submerged in a dilute sulfuric acid solution. The reactions occurring at the anode and cathode are respectively

$$Pb(s) + SO_4^{--} \longleftrightarrow PbSO_4 + 2e^- \quad E_{ox} = -0.350\,V \tag{42.35}$$

$$PbO_2 + 4H^+ + SO_4^{--} + 2e^- \longleftrightarrow PbSO_4 + 2H_2O \quad E(red) = 1.690\,V \tag{42.36}$$

The potential between the electrodes is $E_{rdx} = 1.690 - (-0.350) = 2.040$ Volts, and there are normally six cells in one car battery giving it a terminal voltage of about 12.24 Volts. The oxidation potentials for reactions in common batteries are listed in table 7.3.[3]

Reaction	Potential	$Watts/gram$
Lead-Acid Storage battery		
$Pb(s) + SO_4^{--} \longleftrightarrow PbSO_4 + 2e^-$	+0.3588	
$PbO_2 + 4H^+ + SO_4^{--} + 2e^- \longleftrightarrow PbSO_4 + 2H_2O$	+1.6913	0.022
Mercury Cell		
$Zn(s) + 2(OH)^- \longleftrightarrow Zn(OH)_2 + 2e^-$	+1.260	
$HgO(s) + H_2O + 2e^- \longleftrightarrow Hg(\ell) + 2(OH)^-$	+0.0977	
Zinc Silver Oxide cell		
$Zn(s) + 2(OH)^- \longleftrightarrow Zn(OH)_2 + 2e^-$	+1.260	0.18
$2AgO(s) + H_2O + 2e^- \longleftrightarrow Ag_2O(s) + 2(OH)^-$	+0.607	
Nicad battery		
$Cd(s) + 2(OH)^- \longleftrightarrow Cd(OH)_2 + 2e^-$	+0.809	0.033
$NiO_2(s) + 2H_2O + 2e^- \longleftrightarrow Ni(OH)_2(s) + 2(OH)^-$	+0.490	
Air battery		
$Zn(s) + 2(OH)^- \longleftrightarrow Zn(OH)_2 + 2e^-$	+1.249	
$O_2 + 2H_2O + 4e^- \longleftrightarrow 4(OH)^-$	+0.401	
Fuel Cell		
$H_2 + 2(OH)^- \longleftrightarrow 2H_2O + 2e^-$	+0.8277	0.967
$O_2 + 2H_2O + 4e^- \longleftrightarrow 4(OH)^-$	+0.401	

Table 7.3: Anode and Cathode reactions for some common sources of emf.

Problems

158. Given three resistors, $R_1 = 1000\Omega$, $R_2 = 2000\Omega$ and $R_3 = 5000\Omega$, what is the effective resistance when the resistors are connected in series and in parallel? ans. 8000Ω, 588Ω

[3] "Handbook of Chemistry and Physics" 81st edition, CRC Press, Boca Raton, FL (2000) D155-157.

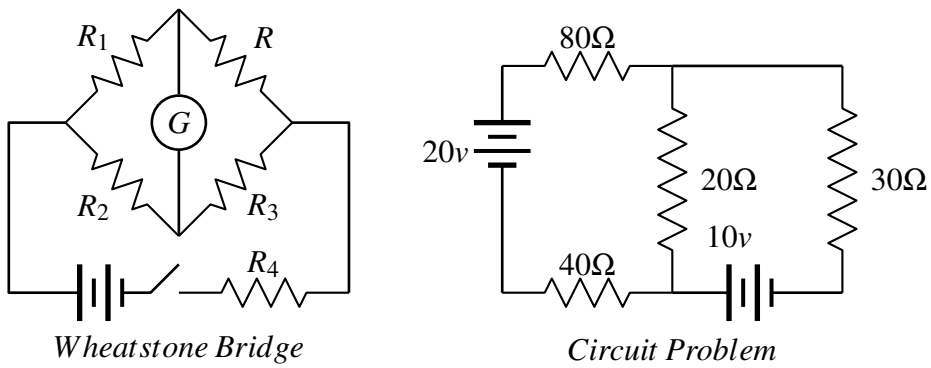


Figure 7.15: DC Circuits

159. The Wheatstone bridge of figure 7.15 is used to measure the value of an unknown resistance R. All resistances are variable except for R. Express the value of R in terms of R_1, R_2 and R_3 after the resistances are adjusted so that no current flows through the galvanometer. ans. $R = R_1R_3/R_2$

160. What is the current flow and the voltage across the resistor and capacitor in a series RC network one microsecond after a voltage of 1 volt is applied across the network. Take R= 1000 ohms and C = 1000 pf. ans. 0.367 milliamp, 0.367 volts, 0.632 volts

161. What is the current flowing in each resistor of the circuit problem in figure 7.15? Each battery has an internal resistance of 2Ω. $I_{80} = 0.178$ amp, $I_{20} = 0.083$ amp, $I_{40} = 0.178$ amp, $I_{30} = 0.261$ amp

162. What is the power loss in each resistor in figure 7.15? $P_{80} = 2.52$ watts, $P_{20} = 0.140$ watts, $P_{40} = 1.26$ watts, $P_{30} = 2.04$ watts

163. What is the approximate emf provided by NiCad batteries, air batteries and fuel cells? ans. 1.30 V, 1.65 V, 1.23 V

Chapter 8

MAGNETOSTATICS

In the chapter on electrostatics, we studied forces and fields arising from stationary electric charges. The consequence was that the charge density ρ the electric field $\vec{\mathbf{E}}$ and the electric displacement $\vec{\mathbf{D}}$ were constant while the current and magnetic field were zero. In magnetostatics, the electric charges will move but if the current $\vec{\mathbf{I}}$ is steady the magnetic field and intensity will also be steady. If the charge density is also maintained constant in the current, the electric field will also be steady. This chapter will be devoted to the study of magnetic forces and fields arising from steady electric currents. At the end of the chapter will also examine the effect of time varying currents.

43 Magnetic Forces and Fields

43.1 Magnetic Forces

The force of one current carrying wire on another was first observed by the Danish physicist and chemist **Hans Christian Oersted** (1777-1851) in 1819 and confirmed by the French physicist and mathematician **Andre Marie Ampere** (1775-1836). Ampere studied this experimentally and found that the force between two current carrying wires varied directly as the magnitude of each current and inversely as the square of the distance between the two current elements. He also found the force depended on the length and orientation of the two wires as well as the medium between the wires. Expressed mathematically these observations become

$$d\vec{\mathbf{F}} = C\frac{I_1 I_2}{r^2}\left(d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \times \hat{\mathbf{r}}_{12}\right) \tag{43.1}$$

In this formula, the constant C depends on the system of units and medium in which the interaction is studied, the vectors, $d\vec{\mathbf{r}}_1$ and $d\vec{\mathbf{r}}_2$ represent segments of wire in each conductor and $\vec{\mathbf{r}}_{12}$ is the distance from one current conducting element to the next.

It is usually convenient to view the force between two current carrying conductors through a field produced by one that acts on the other.

$$d\vec{\mathbf{F}} = I_2 d\vec{\ell}_2 \times d\vec{\mathbf{B}}_1 \quad \textit{where} \tag{43.2a}$$

$$d\vec{\mathbf{B}}_1 = CI_1 \frac{d\vec{\ell}_1 \times \hat{r}_{12}}{r_{12}^2} \tag{43.2b}$$

We will first focus on the magnetic field produced by the first wire, equation 43.2b. This field is produced whether or not the second wire exists. The magnitude of the field may be expressed more simply.

$$dB = CI \frac{d\ell}{r^2} \sin\theta \tag{43.3}$$

where θ is the angle between the wire and the vector r_{12}. The direction of the vector $d\vec{\mathbf{B}}$ is given by the right-hand-rule grasping the conducting wire with the right hand with the thumb pointing in the direction of conventional current flow so that the fingers will point in the direction of the field. This equation is the analog of Coulomb's law for calculating the magnitude of electric fields.

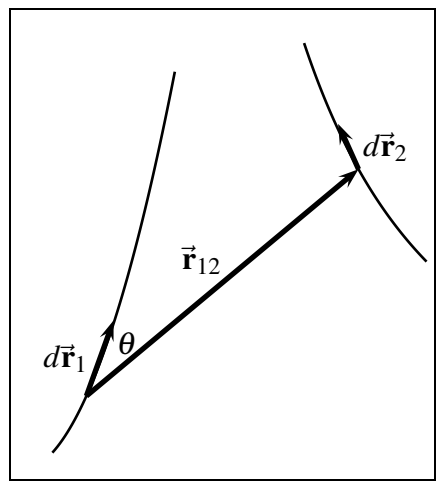


Figure 8.1: Force between two current carrying wires.

Its integral is often called the **Biot-Savart Law**, named after the French physicist **Jean-Baptiste Biot** (1774-1862) who published his work only one month after Oersted and **Felix Savart** (1791-1841) a professor at the College de France and colleague of Biot.

$$B = CI \int_{\ell=-\infty}^{\ell=+\infty} \frac{d\ell}{r^2} \sin\theta \tag{43.4}$$

The terminology and units used in magnetism are often confusing because of the lengthy period of time over which the subject developed. In this text we shall refer to B as the **magnetic field** or **B-field**. It can be seen on setting $F = ma$

that the base units of the magnetic flux are $[B] = kg/A\ sec^2$. In the SI system of units the unit of magnetic flux is the **telsa**, named after **Nikola Tesla**, a prominent Serbian inventor and sometimes competitor of Thomas Edison in the development of electric power. The constant C is normally expressed as

$$C = \frac{\mu_o}{4\pi}. \tag{43.5}$$

The quantity μ_o is called the **permeability of free space** and has the value

$$\mu_o = 4\pi \times 10^{-7}\ N/A^2 \tag{43.6}$$

In the cgs system, the unit **gauss** is used. By definition, a field of one gauss exerts a force, on a conductor, placed in the field of 0.1 dyne per ampere of current per centimeter of conductor. Therefore 1 gauss = $10^{-4} telsa$.

43.2 Magnetic Fields

We will use the Biot-Savart law to calculate the magnetic fields of several current configurations. First consider a long straight wire carrying a steady current I and calculate the magnetic field at a perpendicular distance R from the wire. Using the Biot-Savart law, we can integrate from one end of the wire to the other taking both ends to be infinitely distant from the point at which the perpendicular line bisects the wire.

The angle θ must be taken as the angle between the direction of $d\vec{\mathbf{r}}$ and r. Using $r = \sqrt{(\ell^2 + R^2)}$ and $\sin\theta = R/\sqrt{(\ell^2 + R^2)}$ gives

$$B = \frac{\mu_o I}{4\pi} \int_{\ell=-\infty}^{\ell=+\infty} \frac{\sin\theta}{r^2} d\ell \tag{43.7}$$

$$= \frac{\mu_o I}{4\pi} \int_{\ell=-\infty}^{\ell=+\infty} \frac{R}{(\ell^2 + R^2)^{3/2}} d\ell \tag{43.8}$$

$$= \frac{\mu_o I}{2\pi R} \tag{43.9}$$

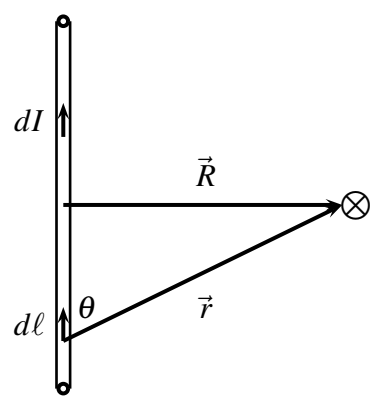


Figure 8.2: Magnetic field of a long, straight current carrying wire.

The circle with the cross sign at the end of the vector R means that, according to the right hand rule, the magnetic field vector points into the page at that point.

The Biot-Savart law can also be used to calculate the magnetic field at a point on the axis of a circular current loop, as illustrated in figure 8.14.

The direction of the magnetic field, $d\vec{B}$, is again given by $d\vec{\ell}_1 \times \vec{r}$ and the right-hand-rule. It can be seen that the perpendicular components of the magnetic field cancel while the axial components reinforce and that the angle between $d\vec{\ell}$ and $\vec{\mathbf{r}}$ is 90 degrees. Therefore

$$B_x = \frac{\mu_o I}{4\pi}\int_{\ell=0}^{\ell=2\pi R}\frac{\sin\phi}{r^2}d\ell \tag{43.10a}$$

$$= \frac{\mu_o I}{4\pi}\int_{\ell=0}^{\ell=2\pi R}\frac{R}{(x^2+R^2)^{3/2}}d\ell \tag{43.10b}$$

$$= \frac{\mu_o}{4\pi}\frac{2I\pi R^2}{(x^2+R^2)^{3/2}} \tag{43.10c}$$

$$= \frac{\mu_o}{4\pi}\frac{2m}{(x^2+R^2)^{3/2}}, \tag{43.10d}$$

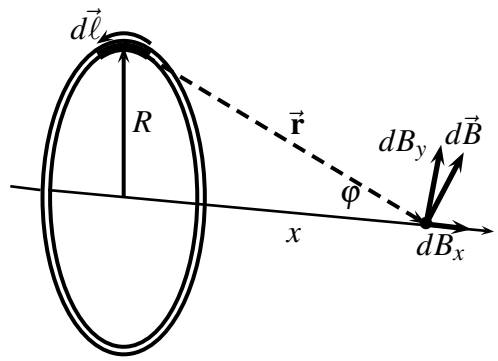


Figure 8.3: Magnetic field of a current loop along axis.

where the quantity m in the numerator is the magnitude of the **magnetic moment** of the current loop

$$m = I\pi R^2, \tag{43.11}$$

which we define as a vector perpendicular to the plane of the loop and in a direction prescribed by the right-hand-rule. At the center of the loop where $x = 0$, equation 43.10d reduces to

$$B = \frac{\mu_o}{4\pi}\frac{2m}{R^3}, \tag{43.12}$$

and at distances $x >> R$, it reduces to

$$B = \frac{\mu_o}{4\pi}\frac{2m}{x^3}. \tag{43.13}$$

This result is similar to the electric field for a dipole along the axis of the dipole, and the magnetic moment of the current loop is similar to the dipole moment of the electric dipole.

If the current consists of a single electron moving in a straight line with velocity $\vec{\mathbf{v}}$, the quantity $Id\vec{\ell}$ can be written as $q\vec{\mathbf{v}}$. This allows us to define the magnetic field that a single moving charge creates at a distance r from its path by

$$\vec{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2} \tag{43.14}$$

If a charge q is moving in a straight line with a velocity $\vec{\mathbf{v}}$ through a magnetic field $\vec{\mathbf{B}}$ making an angle of θ with the direction of the field, the force exerted on the moving charge by the magnetic field is

$$F = q(\vec{\mathbf{v}} \times \vec{\mathbf{B}}). \tag{43.15}$$

The **magnetic flux density** is defined as the integral of the magnetic field strength B over an area

$$\Phi = \oint_A \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}. \tag{43.16}$$

The unit of flux from equation 43.16 in SI units is $Tesla\ meter^2$ which is defined as 1 **weber**.

43.3 Maxwell's Fourth Equation

At this point it is appropriate to compare the magnetic field $\vec{\mathbf{B}}$ to the electric field $\vec{\mathbf{E}}$. In our study of the electric field, we found that the surface integral over any surface enclosing a charge q was equal to the charge a result that was written as Gauss' law in equation 38.1.

$$\varepsilon_o \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = q \tag{43.17}$$

Gauss' law follows from the fact that electric lines of force emanate from from a positive charges and terminate on separate negative charges. We are presented with the situation in magnetostatics where the magnetic field lines terminate on the same current loops that they originate from and thus form closed loops. Therefore, we cannot construct a closed surface around a current loop through which more field lines emerge than enter and the surface integral of the magnetic flux over any surface that encloses a current loop must be zero. This result may be written as Gauss' law for magnetic fields.

$$\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0 \tag{43.18}$$

From the divergence theorem in vector field theory we know that the surface integral of a vector quantity equals the volume integral of the divergence of that vector quantity.

$$\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_V \nabla \cdot \vec{\mathbf{B}} dV \tag{43.19}$$

Since the surface integral of the magnetic flux is zero, we may conclude that the divergence of the magnetic field is also zero.

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{43.20}$$

This result became known as **Maxwell's fourth equation**.

Problems

164. What is the magnetic field at a distance of 10 cm from a long straight wire carrying a current of 6 amperes? ans. 1.2×10^{-5} T

165. What is the magnetic moment of a current loop of 30 cm radius carrying a current of 20 amperes? ans. 5.65 $amp - m^2$

166. What is the magnetic field at the center of the current loop? ans. 4.19×10^{-5} T

167. What is the magnetic field of the current loop at a distance of 1 meter along the center axis of the loop? ans. 1.13×10^{-6} T

168. What is the magnetic field produced by a moving charge at a point on a vector perpendicular to the direction of motion if the charge is a burst of electrons from a linear accelerator carrying 1 microcoulomb of charge with velocity of 1×10^7 m/sec at a point 10 cm from the moving charge? ans. 1.6×10^{-5} T

169. Calculate the magnetic flux passing through a hemisphere perpendicular to the field. ans. $\pi r^2 B$

170. A thin plastic disk or radius R has a charge Q uniformly distributed over its surface and is rotaing at an angular frequency ω about its axis. What is the magnetic field at the center of the disk? ans. $\frac{\mu_o \omega Q}{2\pi R}$

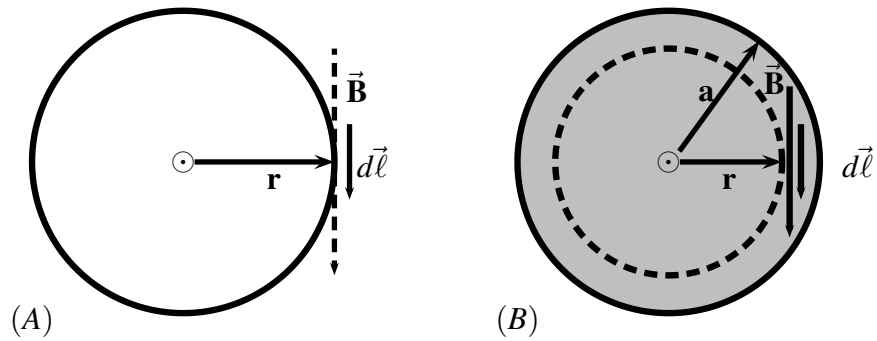


Figure 8.4: Using Ampere's law to calculate magnetic fields outside and inside a long, straight wire.

44 Ampere's Law

In electrostatics we found it convenient to define Gauss' law for use in calculating the magnetic field of symmetric charge distributions. The same case applies in magnetism. We can define **Ampere's law in circuital form** by the line integral around an arbitrary closed loop encircling a current carrying conductor. In this formula, I refers to the net current flowing through the conductor and it is assumed that the wire carrying the current is a long straight wire, a circular loop or some other geometry that does not impose different fields. Consider, for example, the magnetic field of a long straight wire similar to the example in figure 8.4 around which we have constructed a circle of radius r equal to the distance from the wire where we want to calculate the field.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_o I \tag{44.1}$$

The current in the wire is assumed to flow out of the page as denoted by the circle with a dot in the center in figure 8.4(A). The element of circular arc is parallel to the direction of the magnetic field so that $\vec{\mathbf{B}} \cdot d\vec{\ell} = B d\ell$. It is easy to evaluate the line integral around the circle,

$$B(2\pi r) = \mu_o I \tag{44.2a}$$

$$B = \frac{\mu_o I}{2\pi r}. \tag{44.2b}$$

As another example, we can use Ampere's law to find the magnetic field inside a current carrying conductor of radius a where the current density is uniform

throughout the conductor. In this case, we construct a circle about the center of the conductor with a radius r less than the radius of the conductor as illustrated in figure 8.4(B). Using Ampere's law and performing the line integral in the same way and making use of the fact that the portion of the current flow inside the circle of radius r is $I(r/a)^2$,

$$B(2\pi r) = \mu_o I \left(\frac{r}{a}\right)^2 \tag{44.3}$$

$$B = \left(\frac{\mu_o I}{2\pi a^2}\right) r. \tag{44.4}$$

In this case, the magnitude of the magnetic field increases in proportion to the distance from the center of the conductor up to the edge of the conductor. Outside the conductor, the calculation of the previous example applies and the magnetic field decreases with the square of the distance from the center of the conductor.

We can also use Ampere's law to calculate the field inside a solenoid, a long tightly wound helical coil. To apply Ampere's law, we construct a rectangular loop of length L half inside the solenoid and half outside as illustrated in figure 8.5. Inside the solenoid, the field strength is constant in magnitude and direction. Outside the solenoid, the field strength is greatly reduced since the volume outside the solenoid is much greater than the volume inside and all magnetic lines of flux must form a continuous loop. Performing the integral in Ampere's law around the closed path, taking $B = 0$ outside the solenoid and $\vec{\mathbf{B}} \cdot d\vec{\ell} = 0$ on the sides, gives

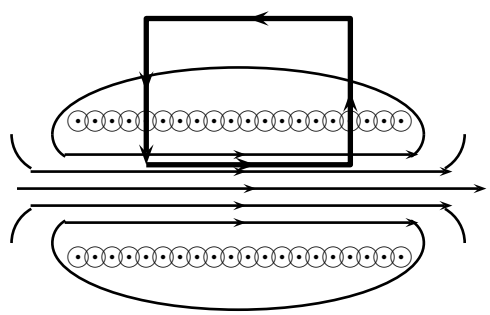


Figure 8.5: Using Ampere's law to calculate the magnetic field inside a solenoid.

$$BL = \mu_o nIL \tag{44.5}$$

$$B = \mu_o nI, \tag{44.6}$$

where n is the number of turns per unit length and I is the current flowing in the coils. From this we see that the magnetic field strength inside the solenoid is proportional to the current and the number of turns per unit length. Since placement

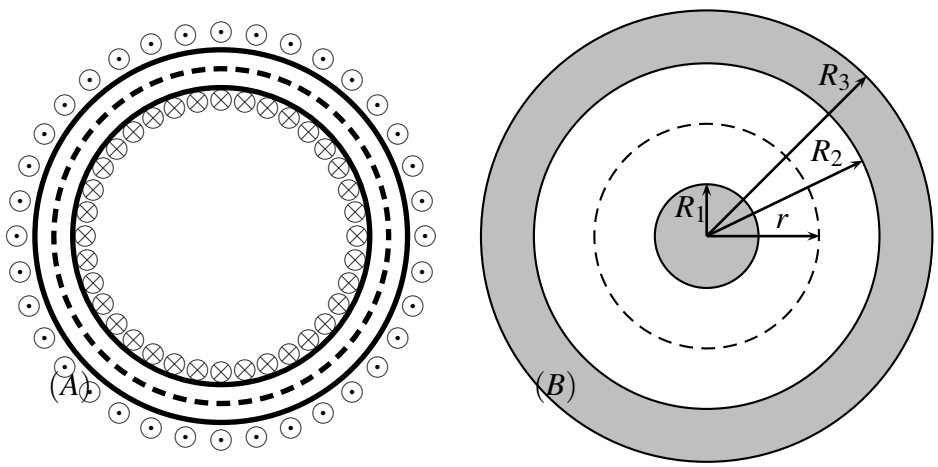


Figure 8.6: Magnetic fields inside toroid and coaxial transmission line.

of the bottom side of the rectangle can be placed anywhere inside the solenoid, it is clear that this field is constant at all positions inside the solenoid.

Inside a toroid, which can be thought of as a solenoid rolled into the shape of a donut with the ends joined, the magnetic lines of flux will be continuous and entirely contained inside the toroid. For this reason the field outside the toroid is zero. To calculate the field inside the toroid we construct a closed loop, represented by a dashed line, inside the toroid as illustrated in figure 8.6(A). Evaluating the integral around the closed loop, we obtain

$$2\pi r B = \mu_o N I L \tag{44.7a}$$

$$B = \frac{\mu_o N I}{2\pi r}, \tag{44.7b}$$

In this case, N is the total number of turns of wire around the toroid. The field B is not constant but depends on the value of r which varies from the inside radius to the outside radius.

As a last example, consider a coaxial transmission line which contains a solid copper rod carrying current in one direction surrounded by a concentric copper cylinder carrying current in the opposite direction. Using Ampere's law, we can find the magnetic field intensity at various points by constructing a circle at each point and performing the line integral as illustrated in figure 8.6(B).

At any point outside the outer conductor, there are no flux lines so the field is zero. Between the two conductors at a distance r from the center of the center

conductor, we can evaluate the integral to get

$$B = \frac{\mu_o I}{2\pi r}, \tag{44.8}$$

showing that the current is dependent only on the current in the center conductor and the radial distance from the center. At a point inside the center conductor, we will obtain the same result as before.

$$B = \frac{\mu_o I}{2\pi R_1^2} r \tag{44.9}$$

Inside the outer conductor the magnetic field is reduced by the fraction of the current flowing inside the selected radius.

$$B = \frac{\mu_o I}{2\pi r} - \frac{\mu_o I}{2\pi r} \frac{\pi\left(r^2 - R_2^2\right)}{\pi\left(R_3^2 - R^3\right)} \tag{44.10}$$

$$= \frac{\mu_o I}{2\pi r} \frac{R_3^2 - r^2}{R_3^2 - R_2^2} \tag{44.11}$$

As a result, we see that the magnetic field strength depends on the radial distance from the center in all parts of the coaxial transmission line except for outside the transmission line where it is zero.

It is interesting to note the effect of a solenoidal field on an electron moving down an evacuated pipe which is wound with wire to form a solenoid. If the electron is moving precisely along the centerline of the pipe the velocity vector of the electron and the field vector of the magnetic field will be parallel, $\vec{\mathbf{v}} \times \vec{\mathbf{B}} = 0$, and there will be no force on the electron. If, however, the electron has a component of its velocity perpendicular to the intended path of motion down the pipe, the cross product $\vec{\mathbf{v}} \times \vec{\mathbf{B}} = vB$ and the magnetic force $F = eB$ exerted perpendicular to both $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ will cause the electron to spiral around its intended path thus keeping the beam from impacting the side of the pipe. This application is used in linear accelerators. The same effect occurs in a betatron or synchrotron which uses a toroid to contain the electrons as they are being accelerated.

Another useful application of Ampere's law is calculation of the flux through a region of space near a current carrying conductor. For example, consider an area of length b and width a with the length positioned a distance d from a wire carrying a current I. The magnetic field at any distance r from the wire is given by equation 44.2(b) so that the flux through the area is

$$\Phi = \int B(r) b dr = \frac{\mu_o I b}{2\pi} \ln \frac{d+a}{d} \tag{44.12}$$

44.1 Maxwell's Second Equation

Ampere's law in circuital form provides a convenient basis to derive Maxwell's second equation. To do this we write the current flowing in a conductor as the surface integral of the current density and use Ampere's law as follows:

$$\oint_{\ell} \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_o \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}}. \tag{44.13}$$

Then converting the line integral to a surface integral using Stoke's law we obtain

$$\oint_S \nabla \times \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_o \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}}, \tag{44.14}$$

which makes it apparent that

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}. \tag{44.15}$$

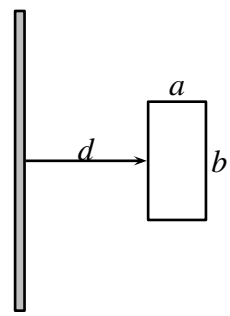


Figure 8.7: Calculating the magnetic flux.

upon replacing $\frac{1}{\mu_o}\vec{\mathbf{B}}$ with $\vec{\mathbf{H}}$. This is Maxwell's second equation in the case that all currents are steady and there is no accumulation of charge, a point that may be emphasized by taking the divergence of both sides and recalling that the divergence of the curl of any vector is zero thus requiring the divergence of the current density J to be zero. Referring back to the continuity equation, equation 42.15, we note that the special case in which the current is steady resulted in a similar requirement in equation 42.16.

There is, however, a problem in that the the charge density is not always constant in a circuit so that $\nabla \cdot \vec{\mathbf{J}}$ is not always zero. Maxwell recognized that he had to add another term to the current density J in equation 44.18 so that

$$\nabla \cdot \left(\vec{\mathbf{J}} + \vec{\mathbf{J}}'\right) = 0 \tag{44.16}$$

Recalling from the continuity equation that $\nabla \cdot \vec{\mathbf{J}}$ = -$\frac{\partial \rho}{\partial t}$ we see it necessary that

$$\nabla \cdot \vec{\mathbf{J}}' = +\frac{\partial \rho}{\partial t} \tag{44.17}$$

Since from equation 41.13 we know that $\nabla \cdot \vec{\mathbf{D}} = \rho$, it is apparent that a suitable definition for $\vec{\mathbf{J}}'$ would be $\frac{\partial D}{\partial t}$ so that equation 44.18 would read

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}. \tag{44.18}$$

This equation became known as Maxwell's second equation and was one of his most important contributions to electromagnetic theory.

Problems

171. What is the magnetic field inside a long straight wire of radius 1mm carrying a current of 10 amperes at a distance of 0.5mm from the center. ans. 0.0025 T

172. What is the magnetic field inside a long solenoid with 10 turns per cm carrying a current of 10 amperes? ans. 0.01256 T

173. What is the magnetic field inside a toroid with 10 turns per cm carrying a current of 10 amperes at a point 1 meter from the center? ans. 0.002 T

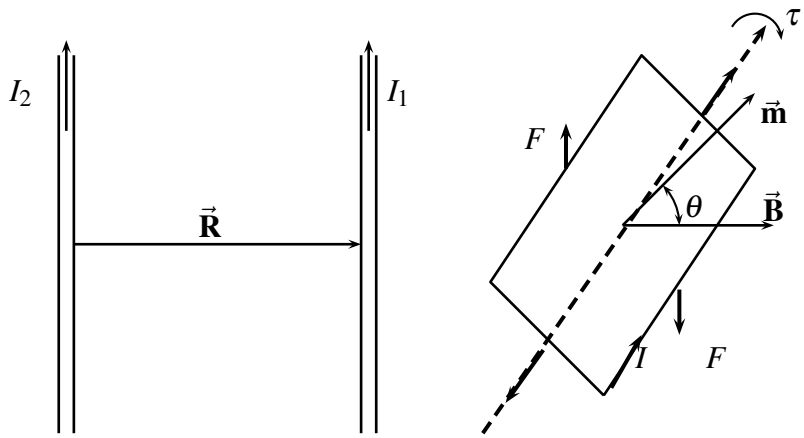
174. Calculate the magnetic flux through a 1 x 2 meter panel when placed immediately below a wire carrying a current of 100 amperes so that the panel is parallel to the wire with its upper side 1 meter from the wire. ans. 2.8×10^{-6} webers

45 Examples of Magnetic Force

Having established means of calculating magnetic forces and fields, we can now investigate the forces exerted by magnetic fields on current loops that are often used to produce electric motors. Consider first the magnetic force between two current carrying conductors a distance R apart as illustrated in figure 8.8(A). The magnetic field of conductor 1 at the position occupied by conductor 2 is $B = \frac{\mu I_1}{2\pi R}$ so that the magnetic force per unit length exerted by conductor 1 on conductor 2 will be

$$F = I_2 B = \frac{\mu_o I_1 I_2}{2\pi R} \tag{45.1}$$

Next, consider a rectangular current loop, of length ℓ, width w and area A, carrying a current I oriented so that the magnetic moment vector $\vec{\mathbf{m}} = I\vec{\mathbf{A}}$ makes an angle θ with the magnetic field lines as illustrated in figure 8.8(B). The force on the lower side, $F = IB\ell$ pulls the wire downward, while a force of the same magnitude on the upper side pulls the wire upward. On the ends, the forces $F = IBw$ are of the same magnitude and cancel. As a result a torque tends to rotate the



(a) Force between two wires (b) Torque on a current loop

Figure 8.8: Force between two wires and torque on rectangular current loop.

loop clockwise about the axis. The force on the lower side, $F = IB\ell$ pulls the wire downward, while a force of the same magnitude on the upper side pulls the wire upward. On the ends, the forces $F = IBw$ are of the same magnitude and cancel. As a result a torque tends to rotate the loop clockwise about the axis.

$$\tau = I(\ell w)B\sin\theta \tag{45.2}$$

or in vector form

$$\tau = \vec{\mathbf{m}} \times \vec{\mathbf{B}} \tag{45.3}$$

By dividing a closed loop of any shape into small rectangular loops for which we have calculated the magnetic moment and torque, it will be seen that the currents on adjacent loops cancel one another so that the magnetic moment of the larger loop is still defined by $\vec{\mathbf{m}} = I\vec{\mathbf{A}}$. Hence the torque is given by equation 45.3, and all that is necessary to know is the current flowing in the loop and the area of the loop.

The magnetic moment of a current loop will be aligned perpendicular to the magnetic field at equilibrium. Then, the work necessary to rotate the loop through an angle θ is obtained from

$$W = \int_{90}^{\theta} \tau d\theta = \int_{90}^{\theta} I(\ell w)B\sin\theta d\theta = -I(\ell w)B\cos\theta = -\vec{\mathbf{m}}\cdot\vec{\mathbf{B}} \tag{45.4}$$

This work will be stored as potential energy of the current loop and is similar to the potential energy of an electric dipole rotated through an angle θ in an electric

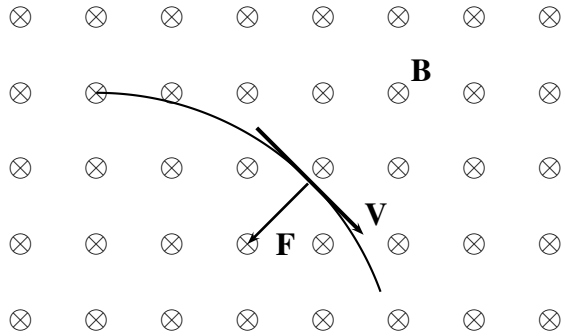


Figure 8.9: Magnetic force on an electron moving through a B-field with velocity V.

field.

$$U = -\vec{\mathbf{p}} \cdot \vec{\mathbf{B}} \tag{45.5}$$

As a final example, consider the force of a magnetic field on a single electron of charge $-e$ moving with a velocity $\vec{\mathbf{V}}$ making a right angle with the magnetic field vector $\vec{\mathbf{B}}$ as illustrated in figure 8.9. In this figure, the magnetic field vector $\vec{\mathbf{B}}$ is represented by a circle with a cross inside to indicate the field is directed into the page. The direction of the cross product of $\vec{\mathbf{V}}$ with $\vec{\mathbf{B}}$ is determined by the right hand rule and will then be directed toward the origin because of the negative sign of the charge.

$$\vec{\mathbf{F}} = -e\vec{\mathbf{V}} \times \vec{\mathbf{B}} \quad \text{and} \tag{45.6}$$

$$F = -eVB \tag{45.7}$$

Since the force is directed toward the origin, the electron will move in a circle with a radius determined by the centrifugal force necessary to balance the magnetic force.

$$\frac{mV^2}{r} = eVB \quad \text{so that} \tag{45.8}$$

$$r = \frac{mV}{eB} \tag{45.9}$$

As a result the radius of the circle is determined by the momentum of the electron mV and the magnetic field B.

Problems

175. What is the force between two wires separated by 1cm and carrying currents of 10 amperes? 0.002 N

176. What is the magnetic moment of a loop of wire with area 1 m^2 carrying a current of 10 amperes? ans. 10 $amp - meter^2$

177. What would be the maximum torque on the current loop in a magnetic field of 0.01 T. ans. 0.1 N-m

178. What amount of work would be necessary to rotate the current loop through 30 degrees? ans. 0.0867 J

179. What force is exerted on a proton moving with velocity 2.00 x 10^7 m/sec through a field of 3.00 w/m^2 at an angle of 30 degrees to the magnetic field? ans. 4.80×10^{-12} N

180. What is the radius of the circle that the proton will move around if its velocity vector is perpendicular to the magnetic flux lines? ans. 6.96 cm

181. If 1% of an electron's momentum is perpendicular to its path of motion inside a linear accelerator when the electron is moving with a velocity of 2.80×10^8 m/sec down a channeling pipe of 3 cm radius that is wound with insulated wire making 10 turns per cm and carrying a current of 10 amperes, what will be the radius of the electron's spiral down the pipe? ans. 1.27 cm

46 Magnetic vector potential

Since the electrostatic scalar potential is so useful in calculating electric fields, it is tempting to define a magnetic potential. There is, however, a problem in defining a magnetic scalar potential. According to Ampere's law, the line integral of the magnetic field around a closed loop $\oint \vec{\mathbf{B}} \cdot d\vec{\ell}$ does not vanish unless the current through the loop is zero; but the line integral of the electric field around a closed loop $\oint \vec{\mathbf{E}} \cdot d\vec{\ell}$ does vanish because the electric field $\vec{\mathbf{E}}$ is a single valued function of position $\vec{\mathbf{E}} = \nabla V$. Any magnetic scalar potential would have to be multiple valued and this would not be satisfactory.

We can, however, define a **magnetic vector potential** that is single valued and will be useful in calculating magnetic fields. To do this we must first recall that the

quantity $\vec{\nabla}$ is a vector quantity and can only operate on another vector by means of a scalar product to form the divergence of that vector ($\vec{\nabla}\cdot A$) or by means of a cross product to form the curl of that vector ($\vec{\nabla}\times A$); and it can operate on a scalar quantity V to form the gradient of that function ($\vec{\nabla}V$). For example,

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2} \tag{46.1}$$

so that the Biot-Savart law may be written in the form

$$\vec{\mathbf{B}} = \frac{\mu_o I}{4\pi}\oint \frac{d\vec{\ell}\times\hat{\mathbf{r}}}{r^2} \tag{46.2}$$

$$= \frac{\mu_o I}{4\pi}\oint \vec{\nabla}\left(\frac{1}{r}\right)\times d\vec{\ell}. \tag{46.3}$$

Now, we can make use of the vector identity

$$\vec{\nabla}\times(VA) = (\vec{\nabla}V)\times A + V(\vec{\nabla}\times A) \tag{46.4}$$

to get

$$\vec{\nabla}\left(\frac{1}{r}\right)\times d\vec{\ell} = \vec{\nabla}\times\left(\frac{d\vec{\ell}}{r}\right) - \left(\frac{1}{r}\right)\left(\vec{\nabla}\times d\vec{\ell}\right). \tag{46.5}$$

Substituting into the expression for $\vec{\mathbf{B}}$ gives

$$B = \frac{\mu_o I}{4\pi}\oint \vec{\nabla}\times\left(\frac{d\vec{\ell}}{r}\right) - \frac{\mu_o I}{4\pi}\oint\left(\frac{1}{r}\right)\left(\vec{\nabla}\times d\vec{\ell}\right). \tag{46.6}$$

The second integral is zero since $\vec{\nabla}$ is a function of the coordinates at the point where $\vec{\mathbf{B}}$ is being calculated and $d\vec{\ell}$ is a function of the coordinates along the closed loop. Therefore, the expression for $\vec{\mathbf{B}}$ can be rewritten to get

$$B = \frac{\mu_o I}{4\pi}\oint \vec{\nabla}\times\left(\frac{d\vec{\ell}}{r}\right) = \vec{\nabla}\times\left[\frac{\mu_o I}{4\pi}\oint\frac{d\vec{\ell}}{r}\right] \tag{46.7}$$

or asnn.

$$B = \vec{\nabla}\times\vec{\mathbf{A}} \quad \text{where} \tag{46.8a}$$

$$\vec{\mathbf{A}} = \frac{\mu_o I}{4\pi}\oint\frac{d\vec{\ell}}{r}. \tag{46.8b}$$

We define $\vec{\mathbf{A}}$ as the **magnetic vector potential**. It has the units of $\mu_o I$ so that the MKS unit is the weber/meter and the cgs unit is the maxwell/centimeter or gauss-centimeter. The magnetic vector potential does not suffer from the multi-valuedness problem of a magnetic scalar potential. It is also useful in deriving fundamental properties of electromagnetic fields. For example, the fact that $\vec{\mathbf{B}}$ is the curl of another vector leads to the conclusion that its divergence vanishes, which shows that Maxwell's fourth equation follows from the definition of the magnetic vector potential.

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{\mathbf{A}} = 0 \tag{46.9}$$

Calculating the magnetic vector potential involves mathematics complex beyond the scope for which this text is intended. However, the results of calculating $\vec{\mathbf{A}}$ for several cases will be enumerated to complete the story in comparing electric and magnetic forces. The interested reader is referred to a text by W. T. Scott for the details of calculations.[1]

The magnetic vector potential $\vec{\mathbf{A}}$ works nicely for a segment of wire and the potential for a loop comprised of segments of straight wire can be constructed by adding the vectors from each loop to allow construction of the magnetic vector potential for complex circuits. $\vec{\mathbf{A}}$ is parallel to the wire and in the direction of current flow, the divergence of $\vec{\mathbf{A}}$ is zero and its curl gives the magnetic field $\vec{\mathbf{B}}$ which is perpendicular to $\vec{\mathbf{A}}$. However, a problem arises with this solution when the length of the wire is extended indefinitely because the magnitude of $\vec{\mathbf{A}}$ becomes infinite. The way around this problem is the same as that for the electric potential V of a straight wire which is also infinite for an infinitely long wire. By calculating $\vec{\mathbf{A}}$ for two parallel wires carrying current in opposite directions a result similar to that obtained for the electric potential of two infinitely long charged wires is obtained. As in the electrical case, everything works as it should. In fact, letting the length of the wires approach infinity reduces the result to a formula similar to that for the electrical potential difference between two oppositely charged wires in section 39.

$$\vec{\mathbf{A}} = \hat{\mathbf{k}} \frac{\mu_o I}{2\pi} \ln \frac{R_2}{R_1} \tag{46.10}$$

The magnetic vector potential for the current loop is an elliptic integral but can be evaluated and is single-valued. At distances great compared to the radius of the

[1] Scott, W.T. "The Physics of Electricity and Magnetism" , John Wiley & Sons, Inc., New York (1959) pp.282-290.

Current configuration	Magnetic vector potential
straight wire of length 2a a distance R from the center point	$\vec{\mathbf{A}} = \mathbf{k}\left(\frac{\mu_o I}{4\pi}\right) ln \frac{\left(a^2+\sqrt{\left(R^2+a^2\right)}\right)^2}{R^2}$
two parallel segments of length 2a at a perpendicular distance R from the center point	$\vec{\mathbf{A}} = \mathbf{k}\left(\frac{\mu_o I}{4\pi}\right) ln \left[\frac{R_1}{R_2}\frac{\left(a^2+\sqrt{\left(R_1^2+z^2\right)}\right)}{\left(a^2+\sqrt{\left(R_2^2+z^2\right)}\right)}\right]$
loop of wire with radius a at a distance z from the plane of the loop and a distance R from the axis of the loop	$\vec{\mathbf{A}} = \hat{\mathbf{e}}_\phi\left(\frac{\mu_o I}{2\pi}\right)\int_0^\pi \frac{a\cos\phi d\phi}{z^2+a^2+R^2-2aR\cos\phi}$
inside a solenoid with radius a at a distance R from the axis of the solenoid	$\vec{\mathbf{A}} = \hat{\mathbf{e}}_\phi \frac{1}{2}\mu_o nIR$

Table 8.1: Magnetic Vector Potential for different configurations.

loop, the magnetic vector potential can be expressed in terms of the magnetic moment of the loop. The magnetic vector potential of the solenoid is the only example in which an analytical calculation can be made. Taking the curl of $\vec{\mathbf{A}}$ gives the value of the magnetic field found earlier.

Problems

182. Prove that $\nabla\cdot(\nabla\times\vec{\mathbf{C}}) = 0$ where $\vec{\mathbf{C}}$ is any vector thereby justifying use of equation 46.9 to derive Maxwell's fourth law.

183. Starting with the magnetic vector potential for a solenoid in table 8.1, calculate the magnetic flux from $B = \nabla\times\vec{\mathbf{A}}$ and show it is the same as calculated using Ampere's circuital law. Note that $\vec{\mathbf{A}}$ is given in cylindrical coordinates.

184. Starting with the magnetic vector potential for a wire of length 2a in table 8.1, calculate the magnetic flux from $B = \nabla\times\vec{\mathbf{A}}$ letting $a >> R$ and

show it is the same as calculated using Ampere's circuital law. Note that $\vec{\mathbf{A}}$ is given in cylindrical coordinates.

47 Inductance

Oersted's work on the magnetic force between current carrying wires in 1819 led the English physicist Michael Faraday (1791-1867) and the American physicist Joseph Henry (1797-1878) to perform experiments which revealed the relationship between magnetism and electricity in 1831. Using apparatus similar to that depicted in figure 8.10, Faraday found that the magnitude of the emf induced in a closed loop is proportional to the time rate of change of the magnetic flux through the loop. This result became known as **Faraday's law**.

$$\varepsilon = -\frac{d\Phi_m}{dt} \tag{47.1}$$

When the magnetic flux, Φ, is expressed in webers the emf, ε, is expressed in volts. The negative sign indicates that the induced current will flow in a direction so that its magnetic flux will oppose the change in the external field that created it. This law is known as **Lenz's law** after the German physicist Heinrich Friedrich Emil Lenz (1804-1865) who discovered the effect in 1834.

z

47.1 Definitions

In figure 8.10 an emf ε_1 is impressed across the primary coil on the left causing a current I_1 to flow through the coil. Assume that there are N_1 turns in the primary coil and N_2 turns in the secondary coil. A portion of the magnetic flux produced by the primary coil will flow through the secondary coil. Changes in this flux will, according to Faraday's law, induce an emf, ε_2 in the secondary coil and cause a current I_2 to flow in the secondary coil. We will designate the flux flowing through the primary coil by Φ_1 and the flux flowing through the secondary coil by

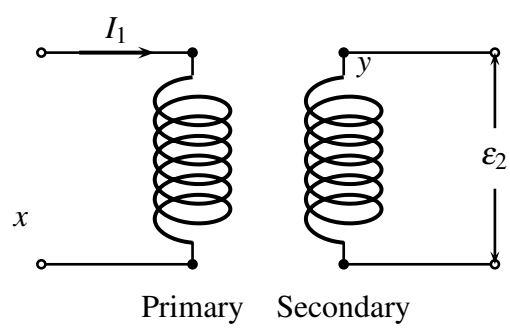


Figure 8.10: Mutual inductance of two coils.

Φ_2. Then according to Faraday's law, the emf induced in the secondary coil will be

$$\varepsilon_2 = -N_2 \frac{d\Phi_2}{dt} \tag{47.2}$$

The negative sign is included to satisfy Lenz's law which states that the emf induced will oppose the change that produced it. Since Φ_2 is generated by I_1 flowing in the primary coil and is proportional to I_1

$$N_2\Phi_2 = M_{21}I_1 \tag{47.3}$$

which allows us to write the emf produced in the secondary coil as proportional to the rate of change of current in the primary coil,

$$\varepsilon_2 = -M_{21}\frac{dI_1}{dt} \tag{47.4}$$

where the constant of proportionality M_{21} is the **mutual inductance** between the coils. In a similar manner, a current change in the secondary coil produces an emf in the primary coil.

$$\varepsilon_1 = -M_{12}\frac{dI_2}{dt} \tag{47.5}$$

The unit of mutual inductance is known as the **henry** in honor of Joseph Henry for his work in magnetism. *We say that the mutual inductance between two coils is 1 henry if a current changing at the rate of 1 ampere per second in one induces an emf of 1 volt in the other.* Mutual inductance is a property of the two coils taken together without regard to which one is defined as primary and which one is defined as secondary, so that we can drop the subscripts and write

$$M_{12} = M_{21}. \tag{47.6}$$

For the same reasons, an emf will be induced in any coil as a result of changes in the current flowing through that coil and we can write.

$$N\Phi = LI \tag{47.7a}$$

$$\varepsilon_1 = -L\frac{dI_1}{dt} \tag{47.7b}$$

where the constant of proportionality L is the **self-inductance** of the coil. In this case, a coil has a self inductance of 1 henry if a current changing at the rate of 1 ampere per second induces an emf of 1 volt in the coil. Lenz's law dictates

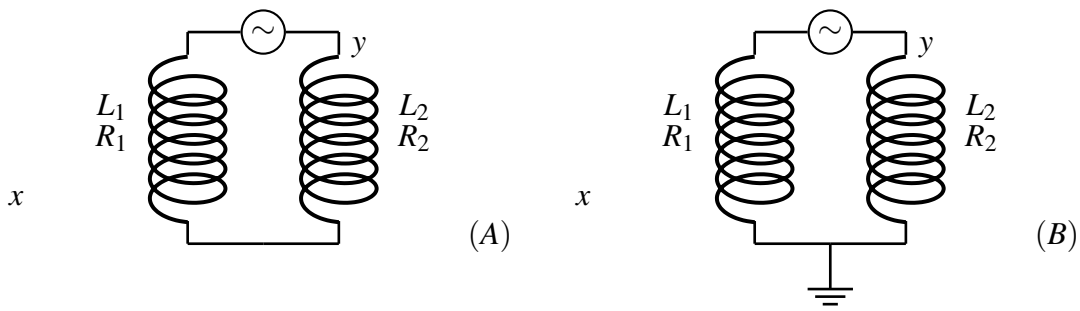


Figure 8.11: Coils in series and parallel.

that the emf induced in a coil because of its own self inductance will be of sign opposite that which produced the change in current and is known as counter-emf or **back-emf**. The **Henry** is a derived unit that may be expressed in terms of the base units of the SI system of units.

$$H = \frac{V \cdot s}{A} = \frac{Wb}{A} = \Omega \cdot s = \frac{J}{A^2} = \frac{kg \cdot m^2}{s^2 \cdot A^2} = \frac{kg \cdot m^2}{C^2} = \frac{T \cdot m^2}{A} = \frac{\Omega}{Hz} \tag{47.8}$$

47.2 Inductors in Series and Parallel

When two coils of inductances L_1 and L_2 and resistances R_1 and R_2 are connected in series as illustrated in figure 8.11(A) with an emf ε impressed across the circuit, the current will be changing at the same rate in both coils so that the back emf induced across each coil will be

$$\varepsilon_1 = -L_1 \frac{dI}{dt} \pm M \frac{dI}{dt} \tag{47.9a}$$

$$\varepsilon_2 = -L_2 \frac{dI}{dt} \pm M \frac{dI}{dt}, \tag{47.9b}$$

where M is the mutual inductance between the coils and the sign of M is positive if the coils are wound in the same direction and negative if they are wound in opposite directions. Kirchoff's law for the circuit can be written as

$$\varepsilon + \varepsilon_1 + \varepsilon_2 = I\left(R_1 + R_2\right) \tag{47.10}$$

$$\varepsilon - \left[L_1 + L_2 \pm 2M\right] \frac{dI}{dt} = I\left(R_1 + R_2\right), \tag{47.11}$$

so that the effective inductance and effective resistance become

$$L = [L_1 + L_2 \pm 2M] \tag{47.12}$$
$$R = R_1 + R_2. \tag{47.13}$$

When two coils of inductances L_1 and L_2 and resistances R_1 and R_2 are connected in parallel as illustrated in figure 8.11(B) with an emf ε impressed across the circuit, the current flowing into the parallel coils will be the sum of the currents flowing into each coil but the emf across each coil will be the same. In this case we can return to equations 47.9 and solve for the currents to get

$$I_1 = \frac{L_2}{L_1L_2 - M^2}\int \varepsilon dt - \frac{M}{L_1L_2 - M^2}\int \varepsilon dt \tag{47.14}$$
$$I_2 = \frac{L_1}{L_1L_2 - M^2}\int \varepsilon dt - \frac{M}{L_1L_2 - M^2}\int \varepsilon dt. \tag{47.15}$$

Adding to get the total current

$$I = I_1 + I_2 = \left[\frac{L_2}{L_1L_2 - M^2} + \frac{L_1}{L_1L_2 - M^2} - \frac{2M}{L_1L_2 - M^2}\right]\int edt \tag{47.16}$$
$$= \frac{L_2 + L_1 - 2M}{L_1L_2 - M^2}\int \varepsilon dt \tag{47.17}$$

we find the effective impedance L for the series combination upon comparing to $I = \frac{1}{L}\int \varepsilon dt$ to be

$$\frac{1}{L} = \frac{L_2 + L_1 - (\pm)2M}{L_1L_2 - M^2} \tag{47.18}$$

The $(\pm)$ sign is included in this equation to signify that the mutual inductance may increase or decrease the effective inductance depending on the direction in which the coils are wound. The interested reader is referred to the Electronic Designers's Handbook for detailed explanation of inductive circuits.[2]

In the event the coils are so placed that the flux from each links all its own

[2]Robert W. Landee, Donovand C. Davis, and Albert P.Albrecht, "Electronic Designer's Handbook" McGraw -Hill Book company, Inc. New York, NY (1957), section 23, page 6

coils with all the coils of the other, we can write

$$L_1 = \frac{N_1\dot{\Phi}_1}{I_1} \tag{47.19}$$

$$L_2 = \frac{N_2\dot{\Phi}_2}{I_2} \tag{47.20}$$

$$M = \frac{N_1\dot{\Phi}_2}{I_2} = \frac{N_2\dot{\Phi}_1}{I_1} \tag{47.21}$$

$$M^2 = \frac{N_1\dot{\Phi}_2}{I_2}\frac{N_2\dot{\Phi}_1}{I_1} = L_1L_2 \quad \text{and} \tag{47.22}$$

$$M = \sqrt{L_1L_2} \tag{47.23}$$

If the coils are identical so that $L_1 = L_2$, it is apparent from the above that

$$M = L_1 = L_2 \quad \text{and} \tag{47.24}$$

$$2M = L_1 + L_2. \tag{47.25}$$

47.3 Calculating Inductance

One method used to calculate the self inductance of coils having certain configurations by calculating the magnetic flux passing through the coil and using equation 47.7(a) to obtain the inductance. For example, suppose the coil is in the form of a solenoid of length ℓ and area A. Since the magnetic field $B = \mu_o nI$ is constant inside the solenoid and the magnetic flux $\Phi = BA$ we can substitute in equation 47.7(a) to get the self inductance of a solenoid

$$N\Phi = LI \tag{47.26}$$

$$n\ell BA = n\ell\mu_o nIA = LI \quad \text{so that} \tag{47.27}$$

$$L = \mu_o n^2 \ell A. \tag{47.28}$$

This result shows that the self inductance of a solenoid of length ℓ and area A is proportional to the volume ℓA. We can also perform the same operation for a toroid. In case of a toroid, however, the magnetic field varies with distance from the center of the toroid so we must integrate over the area to get the flux. Consider a toroid of inner radius a and outer radius b and width h so that the element of

area perpendicular to the magnetic flux is hdr.

$$\Phi = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \int_a^b Bhdr \tag{47.29}$$

$$= \int_a^b \left(\frac{\mu_o NI}{2\pi r} \right) hdr \tag{47.30}$$

$$= \frac{\mu_o NIh}{2\pi} \ln \frac{b}{a} \tag{47.31}$$

Making the same substitutions then gives the self inductance of the toroid to be

$$L = \frac{N\Phi}{I} = \frac{\mu_o N^2 h}{2\pi} \ln \frac{b}{a}. \tag{47.32}$$

47.4 Energy Storage

Since an increase in current through a coil is opposed by a back emf, ε_b, the externally impressed emf must do work during the time that the current rises to a steady state value.

$$W = -\int_0^t i\varepsilon_b dt = L\int_0^I idt = \frac{1}{2}LI^2 \tag{47.33}$$

This energy may be regarded as the **magnetic energy** of the coil residing in the magnetic flux, or in the space pervaded by the magnetic flux. When the current through the coil that sustains the magnetic flux stops flowing, the magnetic flux field collapses releasing the magnetic energy. An emf is induced in the coil by the collapsing field to oppose its collapse thereby transferring energy to the electrons in the coil and inducing an oppositely directed current.

At this point it is appropriate to recall from section 40.4 that the elecgtric energy stored in a capacitor could be represented by $U = \frac{1}{2}\varepsilon_o E^2$. It is easy to derive a very similar equation for the magnetic energy stored in the field of an inductor using the results of the last section. Substituting the inductance of a solenoid from equation **??**(c) into equation 47.33 gives for the energy density inside the solenoid

$$u = \frac{B^2}{2\mu}. \tag{47.34}$$

Another method used to calculate the self inductance is to calculate the energy stored in the inductor and use equations 47.33 and 47.34. For example, the

magnetic field between the conductors of a coaxial cable, $B = \mu_o I/2\pi r$ from equation 44.8 can be used to calculate the energy stored in the magnetic field inside the cable.

$$u = \frac{\mu_o I^2}{8\pi^2 r^2} \tag{47.35}$$

$$U = \int u dV = \frac{m u_o \ell I^2}{4\pi} \int_a^b \frac{dr}{r} \tag{47.36}$$

$$= \frac{m u_o \ell I^2}{4\pi} \ln \frac{b}{a} \tag{47.37}$$

where we have taken the radius of the inner conductor as a and the inside radius of the outer conductor as b for a cable of length ℓ and assumed that the interior of the coaxial cable is air. Now making use of equation 47.33 gives for the self inductance

$$\frac{1}{2} L I^2 = \frac{m u_o \ell I^2}{4\pi} \ln \frac{b}{a} \tag{47.38}$$

$$L = \frac{m u_o \ell}{2\pi} \ln \frac{b}{a}, \tag{47.39}$$

where the magnetic flux in the center conductor has been ignored and the thickness of the outer conductor is assumed small enough that the flux in the outer conductor can be neglected.

47.5 Self Inductance of Common Transmission Lines

The self inductance for several common transmission networks is provided for reference in table 8.2. In all cases the diameter of the wire is d and the distance between the center of the wire to the adjacent wire, ground plane or outer conductor is D.[3]

Problems

185. A coil has a self inductance of 4 mH and carries a current of 1 ampere. If the current is changing at the rate of 2 amperes per second, what is the back emf induced in the coil? ans. 8 millivolts

[3] Ibid. Page 20-18 to 20-23

Configuration	Inductance	Capacitance	Impedance
	$\mu H/meter$	$\mu\mu F/meter$	ohms
single wire distance D above ground plane	$0.460\log\frac{4D}{d}$	$24.12/\log\frac{4D}{d}$	$138\log\frac{4D}{d}$
Two-wire transmission line	$0.921\log\frac{2D}{d}$	$12.06/\log\frac{\sqrt{2}D}{d}$	$276\log\frac{2D}{d}$
Four-wire transmission line	$0.460\log\frac{\sqrt{2}D}{d}$	$24.1/\log\frac{\sqrt{2}D}{d}$	$138\log\frac{\sqrt{2}D}{d}$
Coaxial transmission line	$0.460\log\frac{D}{d}$	$24.1/\log\frac{D}{d}$	$138\log\frac{D}{d}$

Table 8.2: Impedance of common transmission networks.

186. Two coils are placed near to one another so that their mutual inductance is 0.25 mH. If the current is changing at the rate of 2 amperes per second in one, what is the emf induced in the second? ans. 0.5 millivolts

187. Two coils, one of 4 mH and the other of 9 mH self inductance, are placed end to end so that all the magnetic flux of one passes through the other. What is the mutual inductance of the combination? ans. 6 mH

188. Calculate the self inductance of a solenoid would with 10 turns per cm 2 meters long and 10 cm in diameter? ans. 19.7 mH

189. Calculate the self inductance of a coaxial cable used for signal transmission in a laboratory having an inner conductor of 2 mm diameter and an outer conductor of 6 mm inside diameter and length of 10 meters neglecting the effect of flux in the inner conductor. 2.197 μH

190. How much would the flux in the inner conductor increase the self inductance of the coaxial conductor? ans. 0.500 μH

191. Calculate the mutual inductance of two coils if the first coil is in the form of a toroid of N_1 turns, inner radius a, outer radius b and height h and the second is in the form of a coil of N_2 turns closely wound around the first coil as might be used in a synchrotron or betatron to monitor beam current. ans. $\frac{\mu_o N_1 N_2 h}{2\pi}\ln\frac{b}{a}$

192. Calculate the inductance due to flux linkages in the inner conductor of a coaxial cable using the energy method. ans. $\frac{\mu_o \ell}{8\pi}$

193. Derive the formula for the self inductance of a two wire transmission line in which the radius of each wire is a and the distance between the wires is

d. Assume that the current in each wire flows in opposite directions and compute the inductance by finding the flux between the center of one wire and the middle point between the cables and doubling it. ans. $\frac{\mu_o \ell}{\pi}\left(\frac{1}{4}+\ln\frac{d}{a}\right)$

194. An airplane with metal propeller flies due north with the propeller rotating at 2000 rpm. The propeller is 3 m long and the earth's magnetic field is about 10^{-5} $webers/m^2$. What potential is induced between the tips of the propeller? ans. 2.33 millivolts

48 Magnetic properties of matter

The terminology and units of magnetic fields can be, to say the least, confusing for a beginning student. One way to quickly grasp the difference between magnetic force and magnetic flux fields is to consider the field inside a solenoid alternately filled with air (or vacuum) and a material media. In the case of an air core solenoid, the B-field is expressed by

$$B = \mu_o n I \quad \text{where} \tag{48.1}$$

$$\mu_o = 4\pi \times 10^{-7}\, N/A^2 \tag{48.2}$$

$$n = turns/meter \tag{48.3}$$

$$I = current\ in\ amperes \tag{48.4}$$

In this case, the units of the B-field are

$$[B] = \frac{N}{A^2}\frac{1}{M}\frac{A}{1} = \frac{kg}{C\ sec} = Telsas \tag{48.5}$$

and the magnetic flux inside the solenoid is

$$\Phi = BA = \mu_o n I A \quad \text{with units} \tag{48.6}$$

$$[\Phi] = Telsa\ meter^2 \tag{48.7}$$

In case the solenoid is filled with any medium other than air or a vacuum, the B-field must be expressed as

$$B = \mu n I \tag{48.8}$$

where μ, the magnetic permeability of the medium, can be represented by

$$\mu = \mu_r \mu_o \tag{48.9}$$

In this equation μ_r is known as the **relative permeability** equal to the ratio of μ to μ_o. Materials can respond to an applied magnetic field by creating their own magnetic field. Creation of this field is caused by the tendency of the applied B-field to align the magnetic moment of molecules, atoms and free electrons with the applied field. When the magnetic moment vector of an atom, for example, is aligned with the applied field, it strengthens the field and when aligned oppositely to the field it decreases the applied field. Most materials respond to an applied magnetic field to a greater or lesser extent, and materials can be grouped into three categories according to their magnetic behavior.

- **Dimagnetic materials** produce a magnetic field that opposes the applied field and have values of μ_r slightly less than 1. Dimagnetism occurs only in those elements which have even numbers of electrons and in which all electrons are paired with another electron so that the net magnetic moment of the electrons cancel. Examples include most common materials such as wood, water and salt; most organic compounds such as petroleum and plastics; some gases such as H, N and Ar, and many elements including as Sb, Bi, C, Cu, PB, Hg and Ag. Dimagnetism is generally a very weak effect and occurs only in the presence of externally applied B-field. Diamagnetic materials cause lines of magnetic flux to curve away from the material which may be levitated by an external field. A thin slice of pyrolytic graphite, an unusually strong diamagnetic material, about 1 mm thick can be stably levitated in the magnetic field of neodymium-iron-boron supermagnets at a height of about 1 mm to make an interesting classroom demonstration. Strong magnetic fields have even been made to levitate small animals. Superconductors exclude all magnetic lines of flux and have relative permeability of zero.

- **Paramagnetic materials** produce a magnetization in alignment with the applied field and have values of μ_r slightly greater than 1. Paramagnetism occurs in all atoms with an odd number of electrons or in which there is at least one unpaired electron so that the atom can have a net magnetic moment. Examples include air; liquid and gaseous Oxygen; many pure forms of alkali metals, rare-earth elements and the actinides as well as alloys and salts of this group. When placed in an external magnetic field, paramagnetic materials tend to concentrate the lines of magnetic flux and are attracted toward the stronger part of the field. The effect of paramagnetism is greater than dimagnetism, but both are weak effects and there is no residual magnetism when the external magnetic field is turned off.

- **Ferromagnetic materials** can produce a magnetization in alignment with the applied field with values of μ_r much greater than unity. For example, 99.8% pure in its annealed state can have relative permeabilities up to 5,000; 99.95% pure Iron annealed in Hydrogen can have relative permeabilities up to 200,000; 78 Permalloy up to 100,000; Supermally up to 1,000,000 and mumetals up to 100,000. The relative permeability is dependent upon temperature and other factors such as frequency of the applied field. Above a certain temperature, called the **Curie temperature** ferromagnetic materials become paramagnetic. Iron for example has a Curie temperature of 770 ^oC. While the response of dimagnetic and paramagnetic materials is linear with increases in the external field and return to zero when the the field is removed, the response of ferromagnetic materials is non-linear and exhibits hysteresis as will be discussed below.

The easiest way to understand why a material exhibits certain types of magnetic properties and not others is to consider the effect of an applied magnetic field on the atomic electrons when there is pairing of all electrons. When such materials are put in a magnetic field, the electrons circling the nucleus will experience, in addition to their Coulomb attraction to the nucleus, a Lorentz force from the magnetic field. Depending on which direction the electron is orbiting, this force may increase the centripetal force on the electrons, pulling them in towards the nucleus, or it may decrease the force, pulling them away from the nucleus. This effect systematically increases the orbital magnetic moments that were aligned opposite the field, and decreases the ones aligned parallel to the field as required by Lenz's law. This results in a small bulk magnetic moment, with an opposite direction to the applied field. The effect is weak and the material is said to be dimagnetic. If, on the other hand, there is at least one unpaired electron the atoms of that material will have an orbital magnetic moment that can align with the external magnetic field and increase the field. These materials are called paramagnetic. When the external field is turned off, the random motion of the electrons soon remove any trace of the impressed magnetization and the internal magnetic effects are reduced to zero. Ferromagnetic materials differ from paramagnetic materials in that they have clusters of atoms with aligned magnetic moments and produce a stronger field. These clusters tend to retain their alignment when the field reverses or is removed and are commonly called **permanent magnets**.

We could have expressed the B-field of the solenoid in terms of another field

quantity, which we will call an H-field, defined as follows

$$H = nI \quad \text{with units} \tag{48.10}$$

$$[H] = \frac{A}{M} \tag{48.11}$$

and related to the B-field by

$$B = \mu H \tag{48.12}$$

The H-field has the advantage that it is independent of the material inside the solenoid and depends only upon the number of turns and the current flowing through the coils. The terminology for naming these fields has changed over the years, which is why we labeled them as B- and H-fields. The H-field can be thought of as a magnetizing force which produces the B-field and the B-field can be thought of as a magnetic flux field which exerts a force on moving charges and conductors with currents. It may be noted from $F = qvB$ and $F = ILB$ that the agent of the magnetic force in magnetism is the B-field in contrast to electrostatics where the agent for the electric force is E from $F = qE$. The magnetic flux is related to the magnetizing force by $B = \mu H$ while the electric displacement and electric field are related by $D = \varepsilon E$ so that the roles of B and H are switched relative to those of D and E.

The energy of the magnetic field can be expressed in terms of B and H. Referring back to equation 47.33 for the energy stored in the magnetic field of a coil, which we will consider to be wound around a solenoid filled with material of magnetic permeability μ, and substituting $L = \mu n^2 \ell A$ we find for magnetic energy per unit volume

$$E = \frac{1}{\ell A}\frac{1}{2}LI^2 = \frac{1}{2}(\mu nI)(nI) = \frac{1}{2}BH \tag{48.13}$$

for the energy stored in the magnetic field of a solenoid. Although this equation was derived for a solenoid, the results for any configuration of current loops will be the same. This expression may be compared to the equation for the energy density of an electric field, $\frac{1}{2}DE$, to get the total energy density in an electromagnetic field

$$E = \frac{BH + DE}{2} \tag{48.14}$$

By winding a coil around a toroid and connecting it to a source of emf we can establish a magnetic field H inside the toroid. If the toroid is filled with a magnetic material of permeability μ, the magnetic field inside the toroid will be

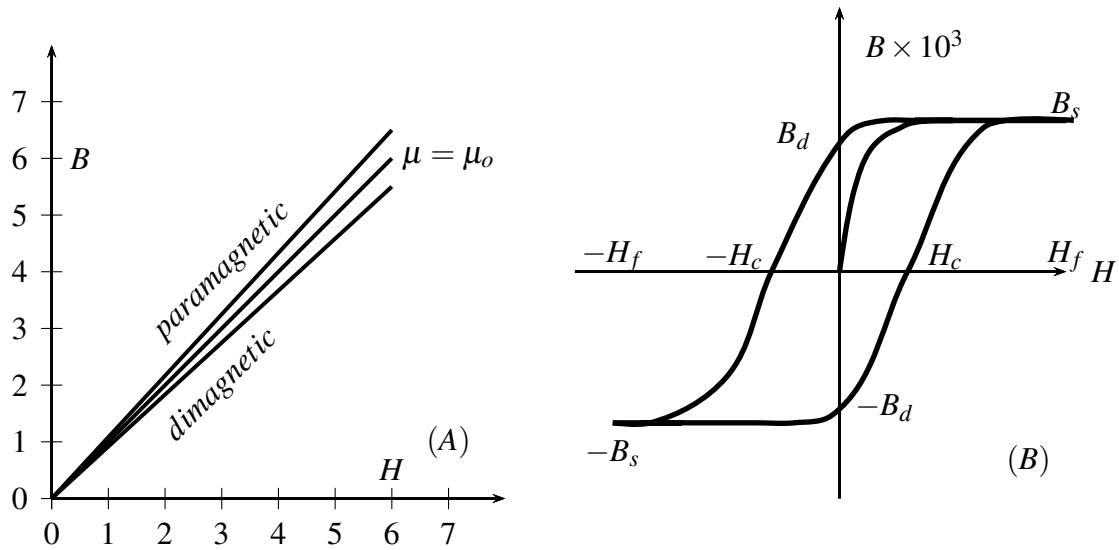


Figure 8.12: B versus H curves for dimagnetic, paramagnetic and ferromagnetic materials.

$B = \mu H$. The B-field can be measured by winding another small coil around the toroid on top of the primary winding and connecting it to a galvanometer. This type of apparatus is used to measure the B and H fields inside a magnetic material. As illustrated in figure 8.12(A), the measured values of B closely follow the impressed field values H for both dimagnetic and paramagnetic materials. The difference between these lines and the line for vacuum in which $B = \mu_o H$ is small, and the lines are linear. The hysteresis curve for a ferromagnetic material is illustrated in figure 8.12(B).

As the experiment begins at at $H = 0$, the B-field B increases until a point is reached where it no longer increases. This value of the B-field B_s is the saturation field at which the magnetic material is said to be **saturated**. As the H-field is decreased the B-field also reduces but the magnetic material retains some of its magnetization when the H-field reaches zero. This value of the B-field B_d is called the **retentivity** of the magnetic material. As the H-field is increased in the opposite direction the B-field finally reaches zero at a value of the H-field $-H_c$ called the **coercive force** of the magnetic material. The B-field will finally reach the saturation level $-B_s$ as the H-field is further increased in the opposite direction to $-H_f$. Then, as the H-field is brought back to zero, the B-field will reach -B_d, then pass through zero at +H_c and return to the saturation level B_s at H_f.

The area enclosed by the hysteresis curve has a special significance. Recalling that the magnetic flux $\Phi = BA$ and that the back emf generated by a change in the flux is $\varepsilon_b = Nd\Phi/dt$, we can calculate the work done against the back emf as the field is changed from B to $B + dB$.

$$dW = -\varepsilon_b I dt = -nAI(A\ell)dB = -HdB(A\ell) \tag{48.15}$$

so that the work per unit volume is $dW = -HdB$. Then the total work per unit volume done in carrying the material completely around the hysteresis loop becomes

$$W = -\oint HdB. \tag{48.16}$$

This work will appear as thermal energy as the material heats up. This is why the temperature of transformer cores increases as more and more current is passed through them.

Problems

195. Show from a completely classical standpoint that an electron orbiting a charge Ze with the outward centrifugal force balanced by the force of Coulomb attraction will have a magnetic moment given by $\frac{e}{2m}L$ where e is the charge, m is the mass and L is the angular momentum of the electron. Calculate the angular momentum and magnetic moment of the orbital electron taking Z=26 and r = $1.2a_o$ where a_o is the radius of the first Bohr orbit for hydrogen. ans. 5.90×10^{-34} J-sec, 5.19×10^{-23} $amp - m^2$ What is the maximum torque that a magnetic field of 5000 T can exert on an atom with this magnetic moment? ans. 2.59×10^{-19} N-m

196. The following data was derived from measurements made at the University of Memphis physics laboratory May 23, 1959, using an iron rod placed inside a solenoid. Plot the hysteresis curve and estimate the saturation vaue, the retentivity and the coercive force of the iron rod. ans. $B_s = 0.85$, $B_d = 0.20$, $H_f = 32500$, $H_c = 1000$

49 Magnetic poles

Magnetism was first encountered in the form of bars of ferromagnetic materials which were observed to be polarized so that the end of one would attract one

H [A/m]	B [T]	H [A/m]	B [T]	H [A/m]	B [T]
9000	0.65	0	0.181	-9000	-0.714
27000	0.736	-9000	-0.663	0	-0.218
52750	0.773	-27000	-0.75	9000	0.63
84250	0.83	-51250	-0.79	27000	0.714
108500	0.845	-84250	-0.845	51250	0.75
82000	0.83	-108500	-0.865	84250	0.796
51250	0.784	-82000	-0.85	108500	0.816
27000	0.745	-51250	-0.805		
9000	0.7	-27000	-0.772		

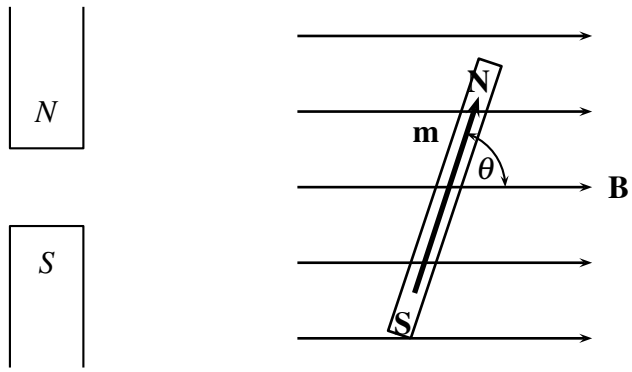
Table 8.3: Hysteresis data

end of another and repel the opposite end. The ends were designated "North" and "South" because it was observed that one end always pointed in a northerly direction toward what was thought to be the earth's north pole. At first it was thought that the magnetic bars were monopoles, but upon cutting the bar in half each half was found to still possess ends of opposite polarity. This resulted in magnetic pole theory being abandoned. We now understand that observations of magnetic fields and forces can be explained in terms of circulating currents. For the sake of historical completeness and because magnetic poles often find a place in modern thinking, the development of magnetic pole theory will be examined.

In magnetic pole theory, we designate each end of a bar magnet as a **magnetic pole** and the use of a formula similar to Coulomb's law for electric charges to describe the force between to bar magnets. This law was discovered experimentally by the French physicist **Charles Augustin de Coulomb** (1736-1806) before a theory of magnetism was developed and the origin of magnetism in ferromagnetic materials was found to be the net effect of atomic currents.

$$F = k\frac{p_1 p_2}{r^2} \tag{49.1}$$

Referring to figure 8.13(a), F is the force exerted by the magnets on one another, p_1 is the pole strength of the magnet 1, p_2 is the pole strength of magnet 2 and r_{12} is the distance between the poles. The force between the two magnets can be thought of in terms of a field created by one that pervades the space between the two magnets and exerts a force on the other. For example, the formula can be rewritten to show B_1 the magnetic field created by magnet 1 and F_{12} the force



(a) Force between bar magnets (b) Magnetic force on a bar magnet immersed in a magnetic field

Figure 8.13: Force between bar magnets and torque on bar magnet immersed in magnetic field

exerted on magnet 2 by magnet 1.

$$\vec{\mathbf{F}}_{12} = p_2 \vec{\mathbf{B}}_1 \quad \text{where} \tag{49.2}$$

$$\vec{\mathbf{B}}_1 = k\frac{p_1}{r_{12}^2}\hat{\mathbf{r}}_{12} \tag{49.3}$$

In the cgs system of units F is measured in dynes and r is measured in centimeters. The original definition of the unit pole strength was such that the constant k in equation 49.1 would be unity. *Thus the unit pole is defined as one which placed 1 centimeter from another like unit pole will repel it with a force of 1 dyne*. The **unit pole**, thus defined, became the basis for the electromagnetic system of units (emu). Following this definition, the ratio of force to pole strength was defined as the **magnetic field intensity** H.

$$H = \frac{F}{p}. \tag{49.4}$$

When F is measured in dynes and p in unit poles, H is expressed in **Oersteds**.

It was found that the torque exerted on a magnet by a uniform magnetic field was proportional to the product of the pole strength m and the distance between the poles d, which gave rise to the definition of the magnetic moment of a bar magnet.

$$\vec{\tau} = \vec{\mathbf{m}} \times \vec{\mathbf{B}} \tag{49.5a}$$

$$\vec{\mathbf{m}} = p\vec{\mathbf{d}} \tag{49.5b}$$

We now understand that the magnetic effects exhibited by a bar magnet are the result of the net magnetic moments of electron orbital motion, electron spin, and nuclear spin in the atoms that comprise the bar magnet so the magnetic moment is more appropriately defined as the product of the circulating current and the area of the current loop. To reconcile this understanding with the magnetic pole theory equations, we can first compare the torque on a current loop illustrated in figure 8.8(b) to the torque on a bar magnet illustrated in figure 8.13(b).

In the case of the torque on the currrent loop in figure 8.8(b) the magnetic moment of the current loop, $\mathbf{m} = IA$, is perpendicular to the plane of the loop and the torque is $\tau = mB\sin\theta$. In this formula, θ is the angle between the vector normal to the plane of the current loop and the direction of the magnetic field. In the case of the torque on the bar magnet in fig:magneticforceandfields(b) the magnetic moment of the bar magnet, $\mathbf{m} = p\mathbf{d}$, is parallel to the axis of the bar magnet and the torque is $\tau = pB\sin\theta$. In this formula, θ is the angle between a vector parallel to the axis of the bar magnet and the direction of the magnetic field. Both equations are of the same format and involve the angle between the moment and the direction of the magnetic field.

What remains is to show that there exists a force between two axial current loops. This is readily shown by Ampere's law for the force between two loops in line integral form.

$$F_{12} = \frac{\mu_o}{4\pi}\int_{C1}\int_{C2}\frac{I_1 d\ell_1 \times (I_2 d\ell_2 \times \hat{r})}{r^3} \tag{49.6}$$

Evaluation of this integral is difficult but the form of the result is evident from figure 8.14 from which the magnetic field along the axis of a current loop was calculated. In this case, it was held that the perpendicular component of the magnetic field, $\vec{\mathbf{B}}_y$ canceled. This, however, is not true for points off the axis. The existence of a radial component of the magnetic field at the circumference of a current loop will result in a force directed toward, or away from, the coil originating the magnetic field depending on the direction of current flow. Evaluation of the integrals in equation 49.6 for a

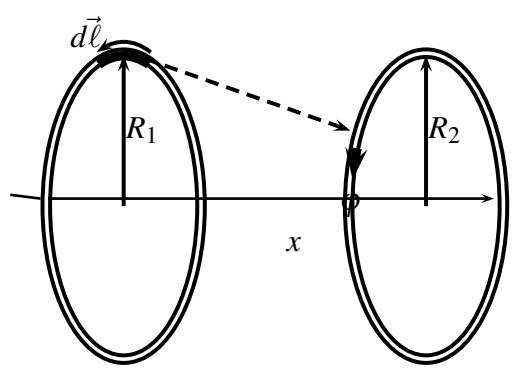


Figure 8.14: Magnetic field of a current loop along axis.

simplified case in which the coils are circu-
lar and their planes are parallel will yield a result of the form

$$F = const\frac{m_1 m_2}{r^2} \tag{49.7}$$

where $\vec{\mathbf{m}}_i$ is the magnetic moment of either loop and r is the distance between their centers. This equation is very similar to equation 49.1 in which the force between two bar magnets was expressed in terms of their pole strengths. Thus the gap between the pole and the circulating current theories is bridged.

Using the definition of the magnetic moment in equation 49.5a, the magnetic moments of several current configurations can be calculated and are listed in the table below.

Current Configuration	Magnetic Moment
bar magnet, length L	$m = pd$
circular loop, area A	$m = IA$
solenoid, N turns, length L	$m = NIA$
charge moving in circle	$m = \frac{1}{2}q\left\|\vec{\mathbf{r}} \times \vec{\mathbf{V}}\right\|$
arbitrary current distribution	$m = \frac{1}{2}\int\left\|\vec{\mathbf{r}} \times \vec{\mathbf{j}}dVol\right\|$

Calculation of the magnetic vector potentials and magnetic fields involves detailed mathematical calculations and will not be pursued further in this text. As an example, however, using equation 46.8(b) to calculate the vector potential and magnetic field a distance r along an off-axis vector for a square current loop of side a gives after higher ordered terms are dropped

$$\vec{\mathbf{A}} \simeq \frac{\mu I a^3}{4\pi r^3}\left[\hat{\mathbf{i}}z - \hat{\mathbf{k}}x\right]. \tag{49.8}$$

From this potential, the Cartesian components of the magnetic field can be calculated.

$$B_x = \frac{3\mu I a^2 xy}{4\pi r^5} \tag{49.9}$$

$$B_y = \frac{\mu I a^2}{4\pi}\left[\frac{2}{r^3} - \frac{3\left(x^2+z^2\right)}{r^5}\right] \tag{49.10}$$

$$B_z = \frac{3\mu I a^2 zy}{4\pi r^5} \tag{49.11}$$

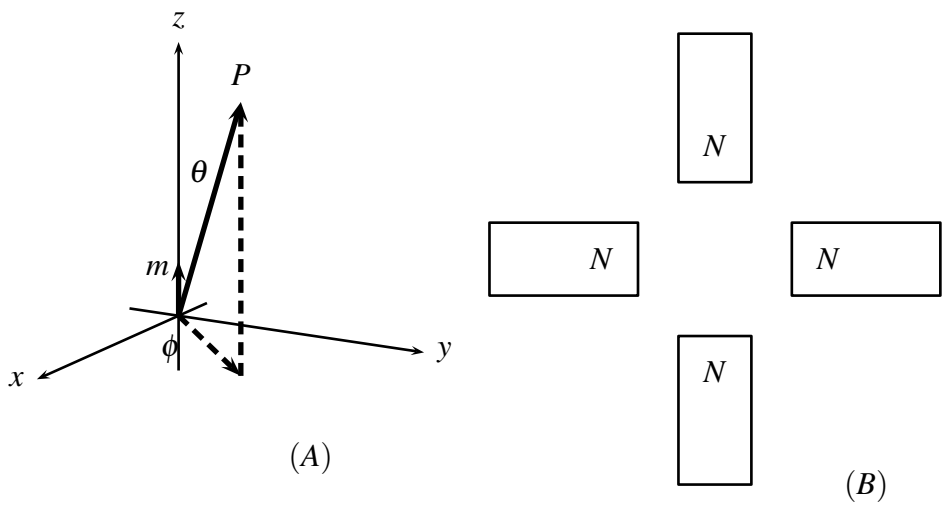


Figure 8.15: Magnetic dipole in polar coordinates (A) and quadrapole magnet (B).

It is often easier to perform calculations in polar coordinates. In this case the expressions for the electric and magnetic fields are quite similar as illustrated in figure 8.15.

$$E = \frac{p}{4\pi\varepsilon r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \tag{49.12}$$

$$B = \frac{\mu m}{4\pi r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \tag{49.13}$$

Another configuration of magnets consists of two groups of magnets positioned so that a pole of one is opposite a pole of the same polarity as illustrated in figure 8.15(B). This is called a quadrapole configuration and creates a magnetic field whose magnitude grows rapidly with the radial distance from its longitudinal axis. As a result, this field can be used to focus and steer charged particle beams in high energy accelerators.

$$E = \frac{p}{4\pi\varepsilon r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \tag{49.14}$$

$$B = \frac{\mu m}{4\pi r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \tag{49.15}$$

Problems

197. What is the magnetic force on a long thin magnet having a pole strength of 5 unit poles when placed in a magnetic field of intensity 50 Oersteds? ans. 250 dynes

198. What is the magnetic force between two circular current loops each of 100 turns of wire and radius 1 cm carrying a current of 10 amperes when they are 10 cm apart with their planes parallel? ans. 0.01256 N

199. What is the magnetic force between two square current loops each of side 1 cm carrying a current of 10 amperes when they are 10 cm apart with their planes parallel? ans. ?????

50 Generating EMF with magnetic fields

A magnetic field may be used to generate an electromotive force by rotating a conducting coil of N turns in the field as illustrated in figure 8.16. In this figure the coil is rotating in a uniform magnetic field in a clockwise direction about the central axis, represented by the dashed line, with an angular velocity of ω. One end of the coil is connected to the large slip ring and the other to the small slip ring. As the coil rotates about the central axis the flux through the coil at any instant is $\Phi = BA\cos\theta$, where B is the magnetic field intensity, A is the area of the coil and $\theta = \omega t$ is the angle between the plane of the coil and the vertical axis. Therefore, the emf induced in the coil is

$$\varepsilon = -\frac{d\Phi}{dt} = NBA\omega\sin\omega t, \tag{50.1}$$

and the simple generator described in figure 8.16(A) produces a sinusoidal alternating emf and will induce an alternating current (or ac current) in a circuit attached to points 1 and 2 as illustrated in figure 8.16(B). The maximum emf induced as the coil is rotated is realized when $\sim \omega t = 1$ and is $\varepsilon_m = NBA\omega$. Since the average value of the sine function over one cycle is $1/\sqrt{2} = 0.707$, the average value of the induced emf becomes

$$\varepsilon_{rms} = \frac{\varepsilon_m}{\sqrt{2}} \tag{50.2}$$

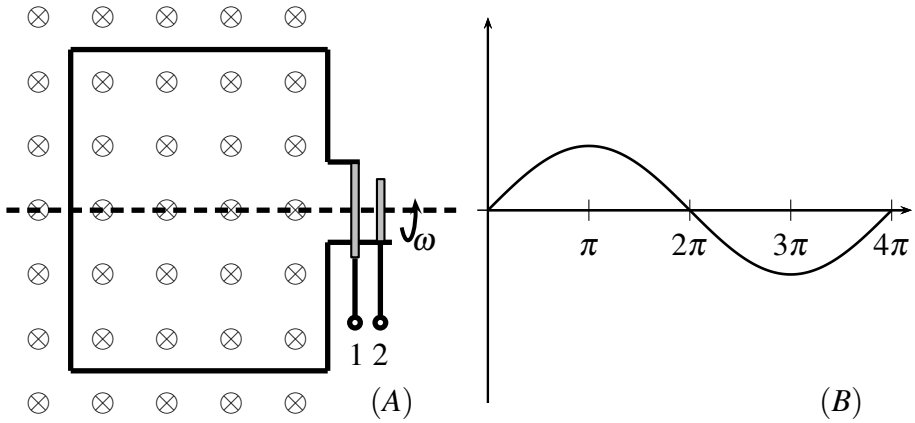


Figure 8.16: Coil rotating in magnetic field (A) and output voltage waveform (B).

This ac generator can be converted to a dc generator by replacing the slip rings with a split commutator causing the connections to the outside circuit to be reversed every time the rotating coil passes through the point where the commutator is split. This would give a rectified output similar to the one pictured in figure 8.17. By adding a second coil the output voltage will be rectified further and by adding additional coils a nearly smooth output can be obtained. These examples are simple representations of how both ac- and dc-generators can be produced by rotating coils in a magnetic field. We will not go further into this but it is apparent that a variety of useful generators can be produced by more sophisticated windings and that these generators can be reversed to produce a motor when a current is driven through the coils from an external power source.

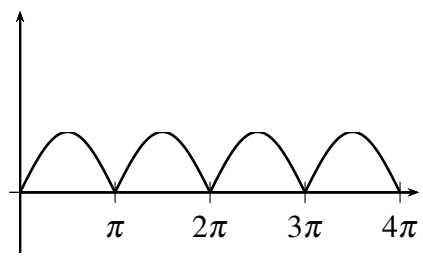


Figure 8.17: Rectified output from ac generator.

Problems

200. The coil of figure 8.16 consists of 200 turns of wire, the area of the coil is 0.05 m^2 and the magnetic flux density is 0.15 tesla. If the coil is being rotated at 1500 rpm, what is the average emf induced across the terminals of the slip rings? ans. 167 volts

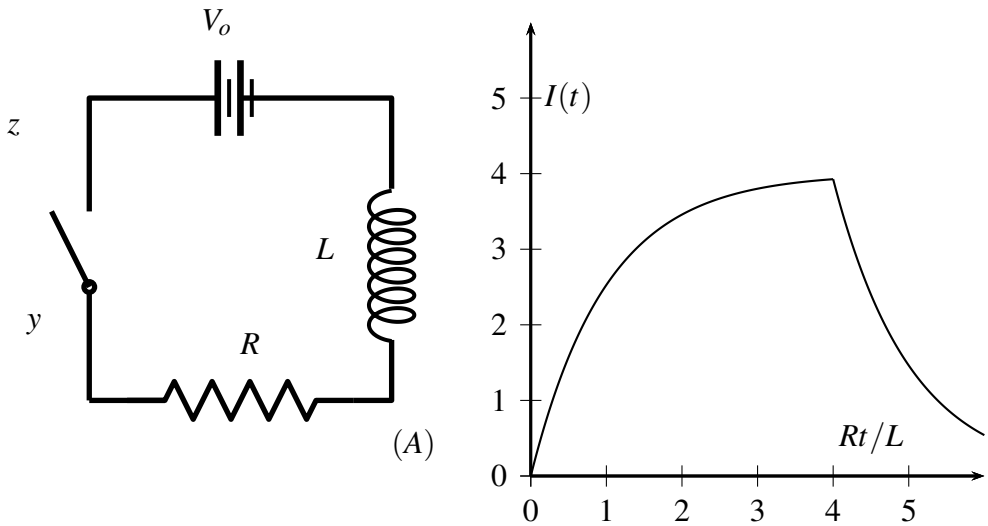


Figure 8.18: Inductor as a circuit element in dc circuits (A) along with change in current after switch is closed and opened(B).

201. An automobile with a straight vertical antenna 2.00 m long is traveling 60 mph when it passes below a d-c power wire carrying a current of 200 amp. If the velocity of the car is parallel to the wire and the tip of the antenna is 2.50 m below the power line, what is the value of the emf induced in the antenna? ans. 0.633 millivolts

51 The inductor as a circuit element

If a coil of inductance L and a resistor R are connected in series with a battery providing an emf V_o and the switch closed as illustrated in figure 8.18(A), the back emf in the coil will retard the buildup of current in the circuit loop. Applying Kirchoff's law to the circuit loop provides

$$V_o - L\frac{di}{dt} = iR. \tag{51.1}$$

This is a first order, linear differential equation that can be solved to obtain an expression for the current in the loop.

$$I(t) = \frac{V_o}{R}\left(1 - e^{-Rt/L}\right) \tag{51.2}$$

The steady state value of the current established after a time long compared to R/L will be $I_o = V_o/r$. The time necessary for the current to build up to a fraction $I = (1-1/e)I_o = 0.6322I_o$ of its steady state value is

$$T = \frac{L}{R}, \tag{51.3}$$

which is called the **time constant** of the circuit. The current in the circuit as a function of time after the switch is closed is often expressed in the form.

$$I(t) = I_o\left(1 - e^{-t/T}\right) \tag{51.4}$$

When the switch is opened, the current will decay toward zero with a the time constant T.

$$I(t) = I_o e^{-t/T} \tag{51.5}$$

This is illustrated in figure 8.18(B) where the current is allowed to build up toward its steady state value for 4 time constants and then the switch is opened.

When the impressed emf is alternating, the results are quite different as illustrated in figure 8.19. In this case the differential equation relating the impressed voltage and current in the loop is

$$\frac{dI}{dt} + \frac{RI}{L} - \frac{V_o}{L}\sin\omega t + \alpha \tag{51.6}$$

which has solution[4]

$$I(t) = -\frac{V_o}{Z}\sin\left[\alpha - \arctan\frac{L\omega}{R}\right]e^{-Rt/L} + \frac{V_o}{Z}\sin\left[(\omega t + \alpha) - \arctan\frac{L\omega}{R}\right] \tag{51.7}$$

where $Z = \sqrt{R^2 + L\omega^2}$ and is the **impedance** to current flow for this circuit. The quantity $X_L = L\omega$ is the **inductive reactance** of this circuit. It is apparent from equation 51.7 that the first term is transient and dies out after several time constants, after which the phase constant α does not contribute anything to the solution. Thus for times greater that several time constants, we can drop the first term and the phase constant alpha and write the solution as

$$I(t) = \frac{V_o}{Z}\sin\left[\omega t - \arctan\frac{X_L}{R}\right] \tag{51.8}$$

[4]Winch, R.P. "Electricity and Magnetism" Printice-Hall, Inc. (1959) p125-132

The current and voltage relationship is illustrated in figure 51.7(B). From this expression, we see that the current lags the voltage by a phase factor $\phi = \arctan \frac{X_L}{R}$. If the resistance of the circuit is zero, the phase factor would be 90°. It is also apparent that the maximum current in the circuit is $\frac{V_o}{Z}$ and that the voltage across the inductor is given by

$$I_o = \frac{V_o}{Z} \tag{51.9}$$

$$V_L = IX_L \tag{51.10}$$

These equations allow the current in the circuit and the voltages across the circuit elements to be easily calculated. For example suppose $r = 50$ ohms, $L = 1$ henry and $V_o = 110$ volts in the circuit of figureeq:inductorcircuitacI(A). In this case

$$X_L = L\omega = 94.2\ ohms \tag{51.11}$$

$$Z = \sqrt{R^2 + X_L^2} = 107\ ohms \tag{51.12}$$

$$I = \frac{V_o}{Z} = 1.03\ amps \tag{51.13}$$

$$V_L = IX_L = 96.8\ volts \tag{51.14}$$

$$V_R = IR = 51.5\ volts \tag{51.15}$$

Problems

202. Suppose that the inductor in a series LR circuit with a battery has an inductance of 2 henrys, the resistor has a value of 150 ohms and the battery provides 150 volts. What is the time constant of the circuit and what will be the value of the current, the voltage across the resistor and the voltage across the inductor at time .010 seconds after closing the switch. ? ans. 0.0133 seconds, 0.53 amperes, 79.3 volts and 70.7 volts.

203. Suppose in a series LR circuit with an ac power source that the impressed voltage is 120 volts, the inductance is 2 henrys, the resistance is 400 ohms and the frequency of the ac voltage is 60 cycles/second. What is the inductive reactance, the impedance, the maximum current, the phase angle, the rms current, the rms voltage and the power loss in the circuit? ans. 754 ohms, 853 ohms, 0.141 amperes, 62 degrees, 0.0997 amperes, 84.7 volts and 3.97 watts.

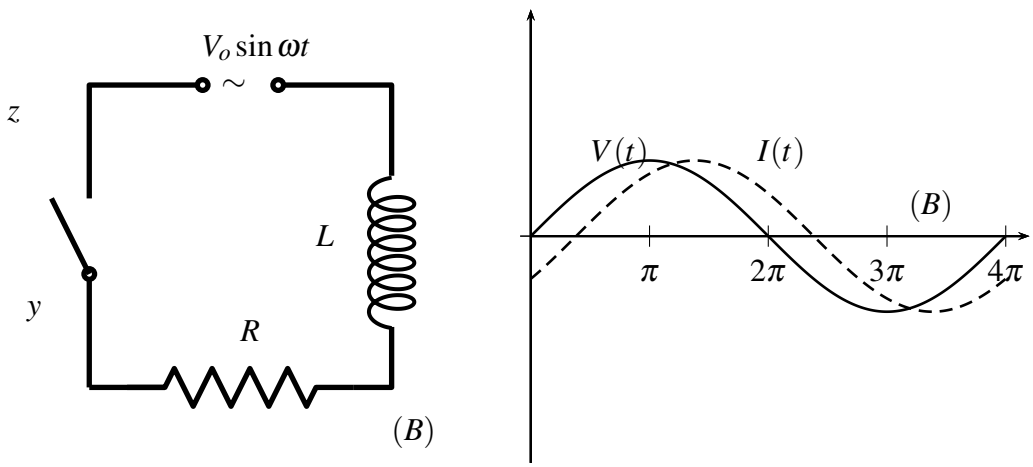


Figure 8.19: Inductor as a circuit element in ac circuits (A) along with current and voltage relationships(B).

52 Summary of magnetic units

Electromagnetism is, by modern day standards, an old science and terminology developed in the earlier periods of study is still in use in several disciplines. As a result, the terminology and units developed can be confusing to the student taking a first course in physics. The following table is provided to help clarify the terminology.

Quantity	*Symbol*	*SIUnit*	*CGSunit*	*English*
Magnetic Induction	B	$tesla(T)$	$gauss(G)$	$Lines/in^2$
Magnetic field strength	H	$\frac{A}{M}$	Oe	$\frac{Amp\,turns}{inch}$
Magnetic flux density	Φ	$Weber(Wb)$	$Maxwell(M)$	$Lines$
Magnetization	M	A/M	emu/cm^3	
Magnetic permeability of vacumn	μ_o	N/A^2	1	$\frac{Lines}{inch\,Amp\,Turn}$
Inductance	L	$Henry$	$Henry$	
Electromotive force	V	$Volt$	$Volt$	
Field force	mmf	$Amp\,turns$	$Gilbert(Gb)$	
Reluctance	R	$\frac{Amp\,turns}{Weber}$	$\frac{Gilberts}{Maxwell}$	$\frac{Amp\,turns}{line}$

Table 8.4: Summary of magnetic units in SI and CGS systems.

Chapter 9

WAVE MOTION AND SOUND

The concept of wave motion is one of the basic tools used to explain physical phenomena. In this chapter, we will introduce wave motion as it is found in vibrating mechanical systems such as long strings and in material media such as solids, liquids and gases. Waves in material media can be of two types (1) shear waves vibrating transversely to the direction of motion or (2) pressure waves vibrating parallel to the direction of motion. We will study both types of waves using the parameters of wave velocity, V_P, wave length, λ, frequency, f or $\omega = 2\pi f$, and group velocity, V_g. The frequency of vibration is controlled by the source causing the vibration and is independent of the medium through which the wave propagates. The wave velocity, however, is a function of the medium through which the wave propagates. The variation of wave velocity with frequency is called dispersion and the relation describing this variation is called the dispersion relation.

53 Transverse Waves in a String

Transverse waves can propagate in both solid and liquid media but generally not in gaseous media. They are produced in numerous ways and in all types of mechanical systems. As a first example to define the parameters needed to describe transverse waves, consider a simple system causing waves to travel down a string as illustrated in figure 9.2. In this system a pushrod attached to a drum rotating with an **angular frequency** ω drives an open ended string. The waves created in the string may be described by

$$y = y_o \cos(kx - \omega t) \tag{53.1}$$

In this equation y_o is the maximum amplitude of the oscillation, x is the position along the x-axis and t is the time at which the wave is viewed. The constant k is the **angular wave number** and ω is the **angular frequency**. The dashed line represents the waveform of the string when the rotating drum is near the bottom of its rotation.

53.1 Wave Parameters

The angular wave number k can be defined in terms of the wavelength λ, which is the distance between similar points on the wave form. In terms of the wavelength, the wave number is the number of oscillations of the wave in a distance 2π.[1] By analogy to the rotating drum in figure 9.2 the quantity ω is the number of radians swept out in one complete cycle, and the **frequency** of oscillation, f, is the number of oscillations per unit time. The **period**, T, can be defined as the time for any point on the string to execute a complete oscillation. These wave parameters are related by

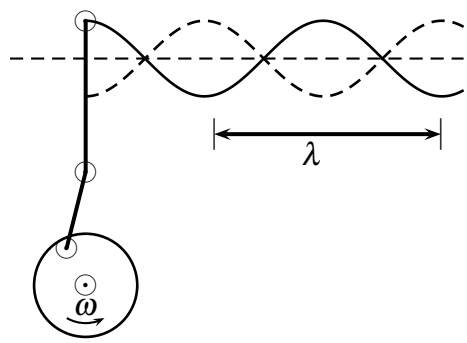


Figure 9.1: Traveling wave in a string.

$$k = \frac{2\pi}{\lambda}, \tag{53.2a}$$

$$\omega = 2\pi f, \quad \text{and} \tag{53.2b}$$

$$T = \frac{2\pi}{\omega}. \tag{53.2c}$$

The string is not moving in the x-direction but the waveform appears to flow past any given point on the x-axis because of the oscillations. The wave velocity, V_P, should be thought of as the velocity in the direction of propagation of any point on the wave form, and does not imply motion of the material media through which the wave is propagating in that direction. To understand this concept better, we can define the argument of equation 53.1 as the **phase** of the wave and regard the wave equation as describing the progression of a condition of constant phase.

[1] Some textbooks define the "wave number" as the number of oscillations per unit distance $k = 1/\lambda$ and the definitions are used interchangeably sometimes leading to confusion.

We can then write this requirement as

$$kx - \omega t = constant \tag{53.3}$$

and differentiate to obtain

$$V_P = \frac{dx}{dt} = \frac{\omega}{k} \tag{53.4}$$

as the **wave velocity**. Thus, the wave velocity is the velocity of any point on the waveform and prompts us to regard equation 53.1 as the equation of a **traveling wave**. Using the wave parameters in equations 53.2, alternative formulas for the wave velocity can be obtained.

$$V_P = \frac{\omega}{k} = \lambda f = \frac{\lambda}{T} \tag{53.5}$$

The parameters V_P, λ, and ω cannot all be independent. Normally, the frequency is fixed by an external condition, such as the frequency of the rotating drum in this example while the phase velocity and wavelength is determined by the medium in oscillation. In the case of a string under tension T undergoing vibrations, the wave velocity may be obtained by approximating a portion of the wave form at its crest by a circle of radius R so that an element of arc length δS can be defined by

$$\delta S = R\theta \tag{53.6}$$

Taking the tension T in the string to be parallel to the string so that the component of the force acting downward at the top of the arc is $2T\sin\frac{\delta\theta}{2}$ and the density of the string to be ρ so that the mass of the element of arc δS will be $\delta m = R\rho\delta\theta$, we can equate the downward force to the centripetal acceleration we obtain

$$2T\sin\frac{\delta\theta}{2} = R\rho\delta\theta\frac{V_P^2}{R}. \tag{53.7}$$

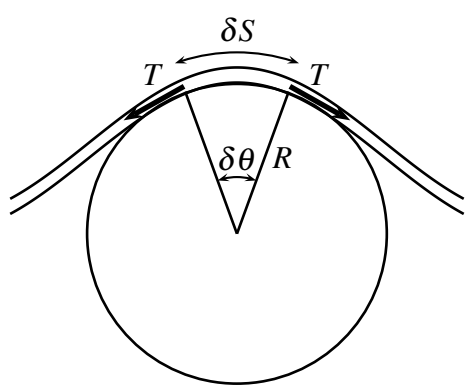


Figure 9.2: Wave velocity in a string under tension.

Assuming $\delta\theta$ small enough that we can approximate the sine of $\delta\theta$ by the angle itself the wave velocity is found to be

$$V_P = \sqrt{\frac{T}{\rho}}. \tag{53.8}$$

In the case of a long string with an open end and not under tension undergoing oscillations small enough so that the only restoring force is Hooke's law, the phase velocity is determined by the shear modulus S and the density of the string.[2]

$$V_P = \sqrt{\frac{S}{\rho}} \tag{53.9}$$

It should be noted that the wave velocity depends only on the medium and not the frequency of the wave motion. Therefore the motion is non-dispersive. The **transverse velocity** and **transverse acceleration** of a particle on the string can be calculated from equation 53.1 by differentiation

$$V_\perp = \frac{dy}{dt} = \omega y_o \sin(kx - \omega t) \tag{53.10}$$

$$a_\perp = \frac{d^2y}{dt^2} = -\omega^2 y_o \cos(kx - \omega t) \tag{53.11}$$

53.2 Intensity

The vertical **velocity** of any point on the waveform executing vertical motion is

$$\frac{dy}{dt} = \omega y_o \sin(kx - \omega t) = \omega\sqrt{y_o^2 - y^2}, \tag{53.12}$$

which has a maximum at $y = 0$ of ωy_o. Thus the **maximum kinetic energy** of any mass element $dm = \rho A d\ell$ in the string will be

$$E_{max} = \frac{1}{2}\rho A d\ell \omega^2 y_o^2, \tag{53.13}$$

so that the **energy density** defined as the maximum energy per unit volume $Ad\ell$ of the string becomes

$$u_d = \frac{1}{2}\rho\omega^2 y_o^2. \tag{53.14}$$

Waves transport energy. The product of the phase velocity and energy density,

$$I = vE_d = \frac{1}{2k}\rho\omega^3 y_o^2, \tag{53.15}$$

is the rate at which energy flows past a given point on the string and is defined as the **intensity** or the **power** of the wave.

[2]Francis A. Jenkins and Harvey E. White "Fundamentals of Optics" McGraw-Hill Book Company, New York (1957) p. 197

53.3 Wave Reflection

The waves thus created will travel down the string until they reach a point where the string is anchored. If the anchor is fixed, such as a string tied to a point on a post, the displacement at the boundary point will remain zero requiring the reflected wave to change its polarity, or stated another way to undergo a 180 degree phase change as illustrated in figure 9.3(A). If the anchor is movable, such as a ring sliding up and down a post, the restoring force on the wave will be zero and there is no control over changing amplitude at the boundary. As a result, the reflected wave is not required to change its polarity or to change its phase as illustrated in figure 9.3(B)

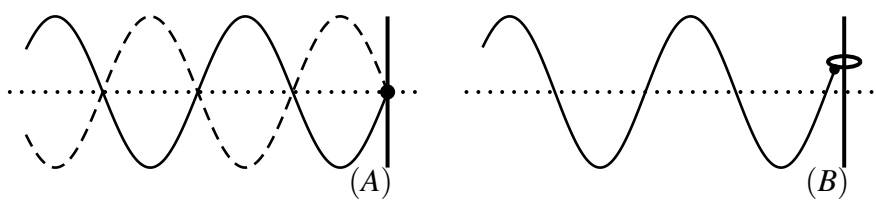


Figure 9.3: Reflection of a transverse string wave at a boundary.

Problems

204. A rope of density 1 kg per m being driven to execute oscillations with an amplitude of 3 cm at a frequency of $f = 3$ cycles/second (Hertz) while under a tension of 2 N. Calculate the phase velocity, wave length, circular wave number, circular frequency, period, energy density and intensity of the waves. ans. $V_p = 1.414$ m/second, $\lambda = 0.47$ meters, $k = 13.33\ m^{-1}$, $\omega =$ 18.84 radians/second, $T = 0.333$ seconds, $u = 0.160\ Joules/m$, $I = 0.226$ $watts$

205. Write the equation of the traveling wave for the last problem.

206. Calculate the transverse velocity and acceleration of a point on the wave 1.5 meters from the origin at 2 seconds. ans. 0.237 m/sec, 9.60 m/sec^2

207. Calculate the maximum transverse velocity and acceleration of any point on the string. ans. 0.566 m/sec, -10.6 m/sec^2

208. Calculate the velocity of transverse waves in bulk Aluminum using shear modulus of 24 GPa and density of 2700 kg/m^3. ans. 2980 m/sec

54 Superposition of waves

Suppose that two waves are propagating through the same medium in the same direction with equal amplitudes y_o and frequencies ω but one is a distance δ ahead of the other. These waves may be represented as follows:

$$y_1 = y_o \sin(kx - \omega t) \tag{54.1}$$
$$y_2 = y_o \sin(kx - \omega t + \delta). \tag{54.2}$$

Superimposing these two waves and making use of the formula for adding trigonometric functions we obtain

$$y = y_1 + y_2 = 2y_o \cos\frac{k\delta}{2} \sin\left[k\left(x + \frac{1}{2}\delta\right) - \omega t\right] \tag{54.3}$$

This sum is a sine wave of the same frequency but amplitude $2y_o \cos\frac{k\delta}{2}$ and shifted along the x-axis a distance $\frac{\delta}{2}$. When $\delta \ll \lambda$, the amplitude is nearly $2y_o$ and nearly 0 when $\delta \sim \frac{1}{2}\lambda$.

A more useful and interesting result is obtained when we add two waves of slightly different wavelengths and frequencies such as for example.

$$y_1 = y_o \sin((k - \delta k)x - (\omega - \delta\omega)t) \tag{54.4}$$
$$y_2 = y_o \sin((k + \delta k)x - (\omega + \delta\omega)t) \tag{54.5}$$
$$y = y_1 + y_2 = 2y_o \sin(kx - \omega t)\cos((\delta k)x - (\delta\omega)t) \quad \text{where} \tag{54.6}$$
$$k = \frac{k_1 + k_2}{2} \quad ; \quad \omega = \frac{\omega_1 + \omega_2}{2} \tag{54.7}$$
$$\delta k = \frac{k_1 - k_2}{2} \quad ; \quad \delta\omega = \frac{\omega_1 - \omega_2}{2} \tag{54.8}$$

The first term in the result is a sine wave that oscillates with the average of the two frequencies. The second term is a modulation term with a longer wavelength determined by δk and a lower frequency determined by $\delta\omega$. The result represents the phenomenon of **beats** between waves close in frequency, useful in tuning musical instruments to match a standardized tuning fork frequency. The time between the beats corresponds to how close the waves are in frequency and how well tuned

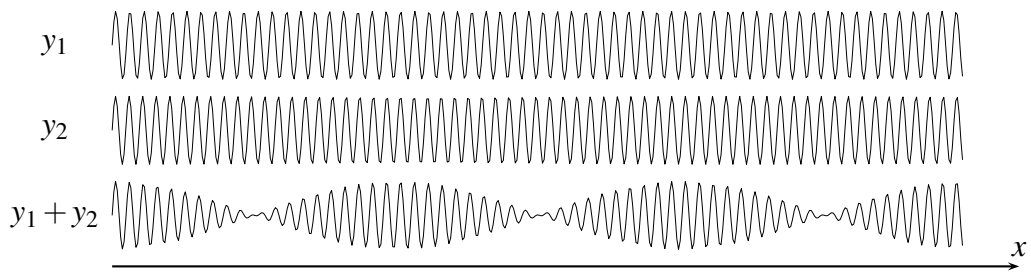


Figure 9.4: Adding two sine waves of slightly different wavelengths and frequencies.

the instrument is. As an example two waves, y_1 and y_2, as described above with $\delta k = 0.05$ and $\delta\omega = 0.05$, are superimposed in figure 9.4.

A more dramatic example is obtained by superimposing five waves of slightly different frequencies and wavelengths as illustrated in figure 9.5.

$$y_1 = 0.49\sin(1.05t - 1.10x) \tag{54.9a}$$
$$y_2 = 0.7\sin(1.025t - 1.05x) \tag{54.9b}$$
$$y_3 = \sin(1.0t - 1.0x) \tag{54.9c}$$
$$y_4 = 0.7\sin(0.975t - 0.95x) \tag{54.9d}$$
$$y_5 = 0.49\sin(0.95t - 0.90x) \tag{54.9e}$$

In this case the grouping of the envelopes is more distinct than when only two waves are combined.

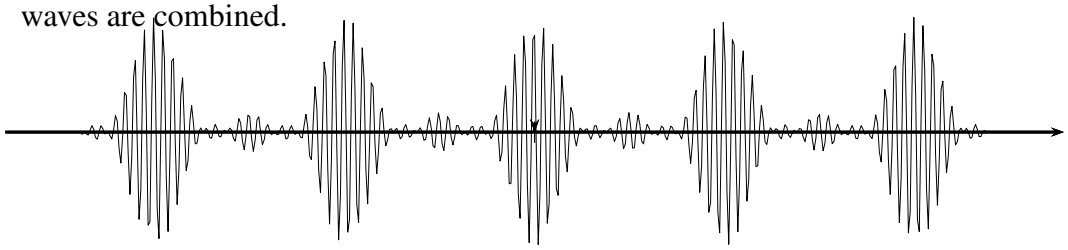


Figure 9.5: Adding sine waves of slightly different wave lengths and frequencies.

54.1 Group Velocity

The wave velocity of the resulting wave is calculated using equation 53.5. The velocity of propagation of the envelope is found using the same formula except

that in the case of the envelope k is replaced by δk and ω by $\delta\omega$. The velocity of the envelope is called the **group velocity**.

$$V_g = \frac{\partial \omega}{\partial k} \tag{54.10}$$

Since the wave velocity V_P is related to the wave number and frequency by $\omega = kV_P$, we can derive a relationship between the group velocity and the wave velocity by differentiating this relation with respect to the wave number.

$$V_g = V_P + k\frac{\partial V_P}{\partial k} \tag{54.11}$$

Expanding one of the wave packets, as illustrated in figure 9.6, allows us to demonstrate the concepts of phase and group velocity more clearly. The wave velocity, designated by V_p, is the speed with which the resulting wave moves along the x-axis. The group velocity, designated by V_g and the dashed line enveloping the resulting wave, is the speed with which the modulating envelope moves along the x-axis. Normally, the group velocity is less than the wave velocity, but in some instances may be greater than the wave velocity.

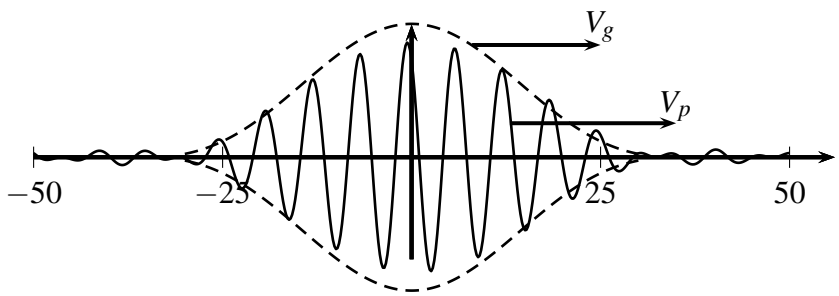


Figure 9.6: One wave packet.

In general, any periodic function may be represented by a sum of sine and cosine functions. A good example of this is the summation of a series of sine waves as indicated in equation 54.12 and figure 9.7 to form a periodic square wave.

$$y = \sum_{n=1,3,5,\ldots} \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right) \tag{54.12}$$

Often in atomic physics a single wave packet is needed to represent a particle. Single wave packets cannot be obtained by the Fourier series approach demonstrated in this section, but single wave packets can be obtained by Fourier integrals in which waves of infinitesimally small difference in frequency are superimposed.

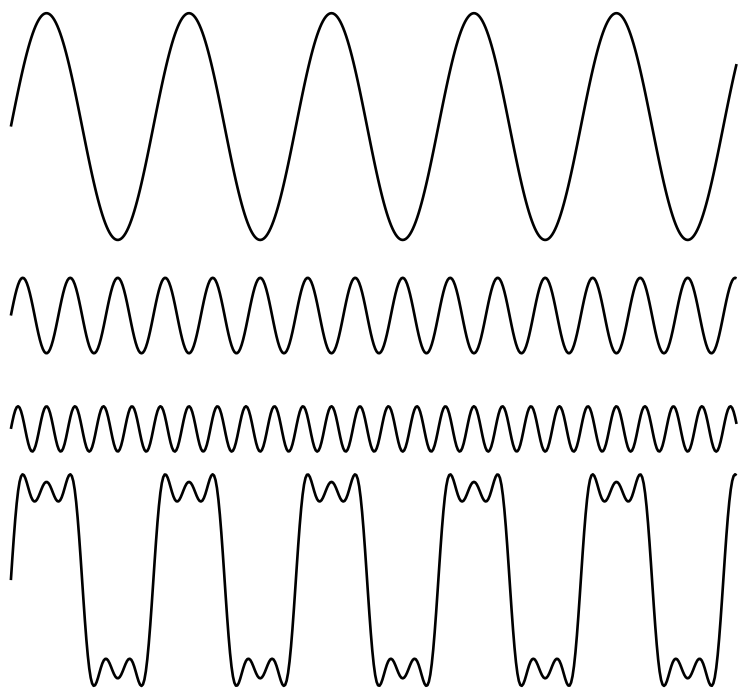


Figure 9.7: Adding sine waves to obtain a periodic square pulse.

54.2 Standing Waves

Another example of wave superposition is the superposition of waves of the same frequency and amplitude moving in opposite directions. Such two waves can be represented by

$$y_1 = y_o \sin(kx - \omega t) \tag{54.13}$$

$$y_2 = y_o \sin(kx + \omega t) \tag{54.14}$$

Superimposing these two waves we obtain[3]

$$y = 2y_o \sin kx \cos \omega t, \tag{54.15}$$

which is the equation of a **standing wave** vibrating in place with frequency ω without exhibiting any movement forward or backward. The standing wave has a

[3]Making use of $\sin\theta_1 + \sin\theta_2 = 2\sin\frac{1}{2}(\theta_1+\theta_2)\cos\frac{1}{2}(\theta_1-\theta_2)$

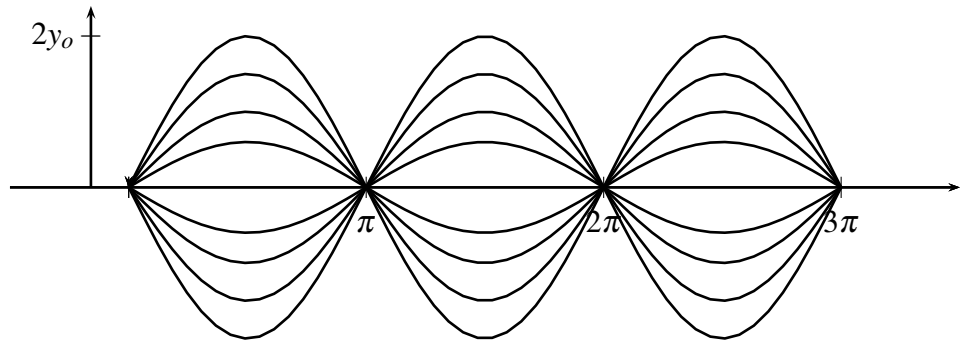


Figure 9.8: Standing wave.

maximum amplitude $2y_o$ at positions where $kx = n\pi/2$, called **antinodes**, and a value of zero at positions where $kx = n\pi$, called **nodes**. No energy is transported by a standing wave because energy cannot pass the nodes which are permanently at rest. Therefore the energy remains "standing" in the vibrating string or other medium.

If the length of the string between the two end nodes is L the number of wavelengths that must fit between the end nodes can be obtained by equating $kx = \frac{2\pi}{\lambda}L = n\pi$ to obtain the wavelength of the standing wave.

$$\lambda = \frac{2L}{n} \tag{54.16}$$

If the string is under tension T the **resonant frequency** of the system is obtained from the relation between the wave velocity, the wave length and frequency found in equation 53.5.

$$f = \frac{n}{2L}\sqrt{\frac{T}{\rho}} \tag{54.17}$$

54.3 Harmonics

The physics of the vibrating string provides a description of the music of string instruments, which are all strings stretched to a given tension. The lowest resonant frequency in equation 54.17 is called the **fundamental** frequency and the

λ	frequency	Overtone	Harmonic
$\frac{2L}{1}$	$\frac{V}{2L}$	Fundamental	First harmonic
$\frac{2L}{2}$	$\frac{2V}{2L}$	First Overtone	Second harmonic
$\frac{2L}{3}$	$\frac{3V}{2L}$	Second Overtone	Third harmonic
$\frac{2L}{4}$	$\frac{4V}{2L}$	Third Overtone	Fourth harmonic
$\frac{2L}{5}$	$\frac{5V}{2L}$	Fourth Overtone	Fifth harmonic
$\frac{2L}{6}$	$\frac{6V}{2L}$	Fifth Overtone	Sixth harmonic

Table 9.1: Harmonic series for string instruments.

others are called **overtones**. The frequency of overtones is always an integral multiple of the fundamental frequency. Together with the fundamental frequency the overtones form a **harmonic series** as outlined in table 9.1.

Several types of string instruments are used in music including the piano, violin, guitar, banjo, mandolin, ukelele and the harpsichord. There three ways of exciting a harmonic series in these instruments, which are striking, bowing or plucking the string. The first method is used in the piano where a hammer is forced against the string when a key is depressed. The second method is used in the violin by dragging a horsehair bow which has a rough surface across the string. The third method of plucking is used for the guitar and the other instruments.

Depending on the method of exciting the string different harmonics will be excited. For example, no harmonic that has a node at the point where the string is plucked can be excited. Plucking the string in the middle will exclude even harmonics with n=1, 2, 3, $\cdots$, etc. all of which have a node at the center of the string. Strings plucked at a point one-third the distance from one end will have harmonics with n=3, 6 and 9, etc. excluded.

There are other ways to start a string vibrating in laboratory experiments. One such method is to attached on end of a string to a tuning fork and pass the other over a pulley. A weight attached to the loose end defines the tension in the string. There will be a node where the string passes over the pulley. Another node will exist at the end attached to the tuning fork although there is a small vibration at this point to allow energy to pass from the tuning fork to the string. If the tension and length of the string are such that one of the harmonics of the string matches the driving frequency of the tuning fork, that harmonic will be strongly excited in the string. This is called **forced resonance**. The amplitude of the vibration will continue to increase until some means of dissipating the energy becomes avail-

able or the string breaks. This same phenomena occurs in mechanical structures of all types, moving and stationary, and is a major problem in the mechanical engineering of these systems.

Problems

209. Sum the first four terms of the series $\frac{1}{n}\sin\frac{n\pi x}{L}$ to obtain a sawtooth wave.

210. Sum the first three odd terms of the series $\frac{1}{n}\sin^2\frac{n\pi x}{L}$ to obtain a series of square pulses of the same polarity.

211. A metal string of mass 2 grams and length 1 meter is under a tension of 50 N. Calculate the phase velocity of a transverse wave, the fundamental frequency and wavelength and the frequency and wavelength of the first overtone and the second overtone. ans. 158/sec, 2 m, 316/sec, 1 m, 474/sec, 0.667 m

212. Sum two waves propagating across the surface of water in mutually perpendicular directions to obtain the wave pattern in the region of interference.

55 Water waves

Surface waves on bodies of water play an important role in everyone's life and have fascinated people for many years. Perhaps the first to study them from a scientific viewpoint was Sir Isaac Newton in 1687. Other early scientists including Faraday 1831, Russell 1844, Euler 1761, Laplace 1774, Lagrange 1781 through more contemporary scientists such as Einstein 1916 have studied and published on them. The subject is a complex one and involves almost all aspects of wave motion. The math is sufficiently involved that a detailed discussion would digress from the stated objective of this book; but the importance is great enough that anyone studying their first course in physics should gain an understanding of the mathematical and scientific basis for the phenomena that we all frequently observe. Our presentation here will be more heuristic than detailed but will address the phenomena that are often observed and reported.

Our first problem is to frame the subject of water waves in a mathematical context. To do this, we can represent the water wave with a Cartesian coordinate system with the x-axis in the direction of flow and take the z-axis as the vertical

axis with $z = 0$ at the average level of the surface and $z = -d$ at the bottom of the lake or ocean containing the water. In the case of a wave moving uniformly in the x-direction with the crest lying in a straight line along the y-axis, the waveform may be thought of as a long cylindrical wave described by a function $z = \xi(x,t)$. We will assume no movement in the y-direction. The motion of every point inside the wave is also described by a velocity vector $\vec{\mathbf{u}}$. Framing the problem in mathematical language is made easy by drawing on the work of early mathematicians. We know that water has little shear force ($S = 0$), that it is incompressible ($\nabla \cdot \vec{\mathbf{u}} = 0$), and irrotational ($\nabla \times \vec{\mathbf{u}} = 0$). Under these conditions we know that the velocity vector is the gradient of some scalar function of position, $\phi(x,y,z,t)$, so that $\vec{\mathbf{u}} = \nabla\phi$). We shall refer to ϕ as the **velocity potential**. Combining this statement with the requirement for incompressibility gives

$$\nabla \cdot \vec{\mathbf{u}} = \nabla \cdot \nabla\phi = \nabla^2\phi = 0, \tag{55.1}$$

which is Laplace's equation for incompressible, irrotational fluids.[4]

The velocity vector has components defining the velocity of flow in all directions. Using the definition of the velocity potential we find

$$u_x = \frac{\partial\phi}{\partial x} \tag{55.2}$$

$$u_y = \frac{\partial\phi}{\partial y} \tag{55.3}$$

$$u_z = \frac{\partial\phi}{\partial z}. \tag{55.4}$$

Laplace's equation contains all the information necessary to characterize the motion of water waves. A general solution to Laplace's equation is

$$\phi(x,z,t) = Ae^{kz}\cos(kx - \omega t) + Be^{-kz}\sin(kx - \omega t), \tag{55.5}$$

which may be verified by substitution. To complete the solution, boundary conditions must be defined. For the first boundary condition, we assume that the vertical velocity of water must be zero at the bottom of the lake or ocean.

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=-d} = 0 \tag{55.6}$$

[4]Refer to section 5.1

For this condition to hold, the constants A and B must be related by

$$B = Ae^{-2kd} \tag{55.7}$$

so that

$$\phi(x,z,t) = A\left[e^{kz} + e^{-k(z+2de)}\right]\cos(kx - \omega t) \tag{55.8}$$

The second boundary condition is obtained at the interface between the water and the atmosphere at a point on the wave. Imagine a volume element of water $dxdydz$ with mass $\rho dxdydz$. The force acting upward will be the differential pressure of the water δP_1 times the cross section of the volume element $\delta P_1 dxdy$, while the forces acting downward will be those of gravity $\rho gzdxdy$ and surface tension $\gamma\kappa dxdy$. This allows us to write

$$P_1 dxdy = \gamma\kappa z dxdy + \rho g dxdy \tag{55.9}$$

The forces of internal pressure and gravity are intuitive but the force due to surface tension requires some discussion. The force of surface tension is the product of the surface tension γ and the curvature κ. For a curve defined by $y = \xi(x)$ the radius of curvature is given by [5]

$$\kappa = \left(\frac{\xi''}{[1+(\xi')^2]^{3/2}}\right), \tag{55.10}$$

which is approximately equal ξ_{xx} for values of $x << a$. If we approximate the arc at the crest of the wave by a circle the equation for the curvature reduces to $\kappa = \xi_{xx}$ for $x = 0$.

In order to use this boundary condition we need to express express each term as derivatives of ϕ. For the upward force $\delta P_1 dxdydz$, we can express the force as a derivative of the momentum of the volume element $F = (dp/dt) = \rho dxdydz(dV_z/dt)$ and find after some algebra that $\rho(dV_z/dt) = -(dP_1/dz)$. The negative sign is used since V_z will be decreasing near the interface. Using $V_z = (\partial\phi/\partial t)$ and integrating results gives $P_1 = -\rho(\partial\phi/\partial t)$. This allows us to write the boundary condition after dividing through by $dxdy$ and setting $z = \xi$ as

$$-\rho\left(\frac{\partial\phi}{\partial t}\right) = \gamma\left(\frac{\partial^2\xi}{\partial x^2}\right) + \rho g\xi \tag{55.11}$$

[5] G.E.F. Sherwood and Angus E. Taylor, "CALCULUS" Prentice-Hall, INC. NJ(1954) page 208.

Differentiating with respect to time gives

$$-\rho\left(\frac{\partial^2\phi}{\partial t^2}\right)=\gamma\left(\frac{\partial^3\xi}{\partial x^2\partial t}\right)+\rho g\left(\frac{\partial\xi}{\partial t}\right) \tag{55.12}$$

Noting that $(\partial\xi/\partial t)=V_z=(\partial\phi/\partial z)$ and rearranging gives

$$\left(\frac{\partial\xi}{\partial t}\right)=-\frac{\gamma}{\rho g}\left(\frac{\partial^3\phi}{\partial x^2\partial z}\right)-\frac{1}{g}\left(\frac{\partial^2\phi}{\partial t^2}\right) \tag{55.13}$$

Applying this constraint to equation 55.8 gives, after some algebra, the relation between the wave parameters λ *and* ω and the water properties g, d *and* ρ.

$$\omega^2=k\left[g+\frac{\gamma}{\rho}k^2\right]\tanh(kd) \tag{55.14}$$

This equation is called the **dispersion relation** and is the basis for explaining the properties of water waves. From it we can obtain an expression for the wave velocity.

$$V_P=\sqrt{\left[\frac{g}{k}+\frac{\gamma k}{\rho}\right]\tanh(kd)} \tag{55.15}$$

$$=\sqrt{\left[\frac{g\lambda}{2\pi}+\frac{2\pi\gamma}{\rho\lambda}\right]\tanh\left(\frac{2d\pi}{\lambda}\right)} \tag{55.16}$$

It is difficult to visualize this relationship because it relates wave velocity, wave length and depth in the same expression. For this reason it is best to break the analysis down in steps. To begin, it should be noted that the hyperbolic tangent, $tanh(kd)$, may be approximated by its argument, kd if the argument is less than about $1/4$ and by unity if its argument is greater than about 3.14. This gives us a relationship between wavelength

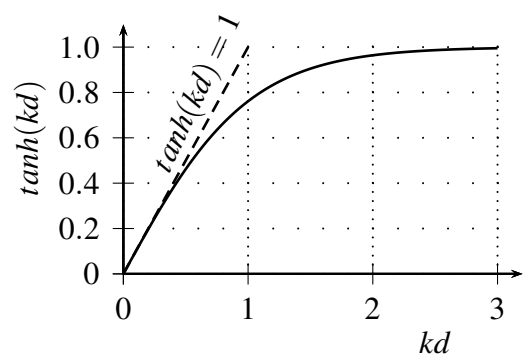


Figure 9.9: Hyperbolic tangent.

and depth of the water where these approximations are valid.

$$d \geq \frac{1}{2}\lambda \quad tanh(kd) = 1 \tag{55.17a}$$

$$d \leq \frac{1}{25}\lambda \quad tanh(kd) = kd \tag{55.17b}$$

55.1 Deep Water Waves

In the first case where the depth is greater than about 1/2 the wave length, waves are generally considered to be **deep water waves**.[6] In this case, the hyperbolic tangent can be dropped from the dispersion relation so that the wave velocity and group velocity of the water waves can be written in a simplified form. These waves are often referred to as **gravity-capillary** waves since the wave velocity is controlled by both gravity and surface tension while depth does not influence wave velocity.

$$V_P = \frac{\omega}{k} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda}} \tag{55.18}$$

$$V_g = \frac{\partial\omega}{\partial k} = \frac{1}{2}\frac{\frac{g}{k} + \frac{3\gamma}{\rho}k}{\sqrt{\frac{g}{k} + \frac{\gamma}{\rho}k}} \tag{55.19}$$

However, these expressions are still not completely intuitive. Equation 55.18 may be reduced to a quadratic and solved using the quadratic formula to obtain

$$\lambda = \frac{V_P^2 \pm \sqrt{V_P^4 - 0.00290}}{3.12}. \tag{55.20}$$

This relation is plotted in figure 9.10 for both short and long wave lengths in deep water. As figure 9.10(A) illustrates, this equation is double valued and has a minimum wave velocity which may be obtained by differentiating equation 55.18

[6]This terminology may appear counter-intuitive at first since we have imposed no restriction on the minimum depth of water for deep water waves. It must be emphasized that the definition applies to the relation between wavelength and depth, not actual depth.

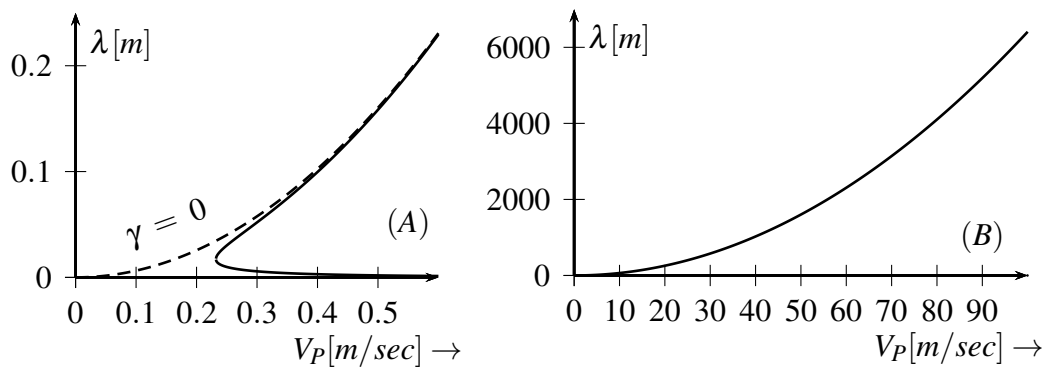


Figure 9.10: Variation of wave velocity with wavelength for short wavelength (A) and long wavelength (B) in deep water waves.

and setting the result equal to zero to obtain

$$\lambda_o = \sqrt{\frac{4\pi^2\gamma}{g\rho}} = 1.74\ cm \tag{55.21}$$

$$V_P(\lambda_o) = \sqrt{2\sqrt{\frac{g\gamma}{\rho}}} = 23.4\ cm/sec \tag{55.22}$$

From this we see that there is a minimum velocity of 23.4 m/sec necessary for wind to create waves. At this point, the waves created will be simply **ripples** in the surface of the water with short wavelengths. It should also be noted that at $\lambda = \lambda_o$ gravity and surface tension contribute equally to the wave velocity and the group velocity equals the wave velocity.

$$V_g = V_P = \sqrt{2\sqrt{\frac{g\gamma}{\rho}}} \tag{55.23}$$

These waves are often referred to as **capillary waves** since surface tension is normally associated with the rise of liquid in a capillary tube. Capillary waves contribute to the roughness of the sea and hydrodynamic drag on small craft. As the wavelength of water waves becomes less than λ_o surface tension dominates in equation 55.18 and the wave velocity can be approximated by

$$V_P = \sqrt{\frac{2\pi\gamma}{\rho\lambda}} \sim \frac{21.6}{\sqrt{\lambda}}\ cm/sec. \tag{55.24}$$

Also, the group velocity trends toward $3/2$ the wave velocity and the waves appear to be running backward through the envelope.

$$V_g = \frac{3}{2}V_P = \frac{3}{2}\sqrt{\frac{2\pi\gamma}{\rho\lambda}} \sim \frac{32.4}{\sqrt{\lambda}}\ cm/sec \tag{55.25}$$

Above λ_o the gravity term begins to take control in equation 55.18 with the surface tension term contributing less than 3% at $\lambda = 0.1$ meter. At the longer wavelengths, the wave velocity may be approximated by

$$V_P = \sqrt{\frac{g\lambda}{2\pi}} \sim 1.25\sqrt{\lambda} \tag{55.26}$$

as the group velocity trends toward one-half the wave velocity so that the waves appear to be running forward through the envelope.

$$V_g = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}} \tag{55.27}$$

It is important to note that depth does not enter any of these "deep water" equations and that wave velocity and group velocity are always controlled by gravity and surface tension. In the case of longer wavelengths, the wave velocity also becomes large as illustrated in figure 9.10(B). For example, at a wave velocity of 40 m/s the wavelength of the waves will be about 1000 meters. Recall that the average depth of the ocean is about 4,000 meters so that these waves still behave as deep water waves and are described by the equations above.

The period of a wave T can be obtained from the wave velocity and wave length. For deep water gravity waves where the wave velocity is given by equation 55.26, there exists a simple relation for the period of the waves.

$$T = \sqrt{\frac{2\pi\lambda}{g}} \quad \text{or} \tag{55.28a}$$

$$V_P = \frac{g}{2\pi}T \tag{55.28b}$$

Before leaving the subject of deep water waves, a short discourse on the origin and development of wind waves in the ocean may prove helpful for further discussion and study. First, we examine the effects of wind waves as observed in ocean going vessels. As wind begins to blow across the surface of the water it begins to

gather water and create pressure and stress in the water surface. These factors create small waves with wavelength less than 1.74 cm and velocities a little greater than 23 m/sec, which are commonly called **ripples** or **capillary** waves. Surface tension of water is the dominant force controlling wavelength and velocity. As the wind continues to blow it gathers more and more water and transfers more and more energy to the waves driving the waves faster and higher until gravity takes over as the dominant force. These waves are called **gravity** waves. Three factors determine the amount of energy that wind transfers to water waves: (1) wind speed, (2) time the wind blows in one direction and (3) distance over which the wind continues to blow, called the **fetch** of the wind. As indicated by the dispersion equation, waves with longer wavelengths travel faster than waves with shorter wavelengths. As the wind continues to blow and the fetch increases the wave can no longer support the weight of water and the crest begins to **break** forming **white caps**. As the waves near this condition, the sea is said to be **fully developed**.[7]

Typical ocean waves may have crest heights of about 2 meters, wavelengths of 35 meters and periods of 10 seconds at wind velocities of 10 m/sec. This is sometimes referred to as a "moderate sea". A 40 mph wind blowing for 40 hours with a fetch of 800 miles might be expected to produce waves with a height of 8 meters, a wavelength of 140 meters and a period of 12 seconds. One of the largest ocean waves was documented in 1935 by the USS Ramapo and reported to have a crest height of 34 meters, a wavelength of 350 meters and a period of 15 seconds at wind speeds of 67 mph.

55.2 Shallow Water Waves

In the limiting case described in equation 55.17(b), where the depth of the water is less that $\frac{1}{25}$ that of the wave length water waves are generally referred to as **shallow water waves**. In this case, the hyperbolic tangent may be replaced by its argument to obtain for the phase and group velocities

$$V_P = \frac{\omega}{k} = \sqrt{gd + \frac{4\pi^2\gamma d}{\rho\lambda^2}} \simeq \sqrt{gd} \tag{55.29a}$$

$$V_g = V_P + \frac{k^2\gamma d}{\rho V_P} \tag{55.29b}$$

[7] An often quoted rule of thumb for a fully developed sea is a ratio of wavelength to height of the wave crest of 1:7 and angle of the crest of about 120 degrees.

As a result, shallow water waves are non dispersive and the wave velocity is controlled by the depth. From these equations, we can explain several commonly observed phenomena. One phenomena is that of **surf**. Ocean waves coming ashore transition from deep water waves to shallow water waves and slow down as the depth of the water decreases. For example, a deep water ocean wave from a moderate sea moving at 10 m/sec will slow down to $V_P = \sqrt{gd} = 3.13$ m/sec at water depths of 1 meter. Their period remains the same causing the wavelength to decrease with the wave velocity. The result is that the wave crests become more narrow and increase in height. As their width decreases and height increases water waves become unable to support the mass of water and break when the water depth becomes less than about 1.3 times their height. Waves from a calm sea are normally observed to break at wave heights of 1 to 2 meters as they come ashore; but waves from a 60-knot storm at sea with heights of 7 meters and a period of 20 seconds have been observed to crest at 16 meters coming ashore. In general, the shape of waves as they come ashore is determined by their wavelength while breaking is determined by water depth. There are several types of breaking waves. **Spilling breakers** result from waves of low steepness (H/λ ratio) breaking on gently sloping beaches; **plunging breakers** result from steeper waves breaking over moderate slopes and **surging breakers** result when the beach slope exceeds wave steepness. Surfers find spilling breakers easier to handle but less interesting that plunging breakers. Surging breakers can be dangerous.

Tides result from the gravitational pull of the moon and the sun and centrifugal force of rotation as well as numerous other factors. Tides are actually shallow water waves as may be seen by considering an ideal tide to have a wavelength of 20,000 km, about half the earth's circumference. The average depth of the ocean, about 4.7 km in the Pacific or 3.9 km in the Atlantic, is considerably less than 1/25 of this wavelength equal to about 800 km; therefore tides may be considered to be shallow water waves. Taking the period of a semidiurnal lunar time to be 12.42 hours, the resulting wave velocity of a tide would then be $V_P = \lambda/T = 447$ meters/second. However, the wavelength of the tide is greater than the ocean depth so friction reduces this velocity to less than half the calculated value. The force of gravity from the moon is the largest amounting to $1.1 \times 10^{-7} g$, while that of the sun is $0.52 \times 10^{-7} g$ about 47% that of the moon. The average height of a tide wave in the ocean will be somewhat less than 1 meter, but this is still enough to store a huge amount of energy making it possible to harness energy from tide waves to power cities and towns. Tides are normally calm and regular in their appearance and should not be confused with tidal waves.

Tsunamis, sometimes called **tidal waves**, are generated by earthquakes and

volcanoes in the ocean and sometimes landslides. These waves may have wavelengths ranging from 200 to 1000 km. Because of their large wavelength, tsunamis are also classed as shallow water waves. Their wave velocity might therefore be predicted from $V_P = \sqrt{gd} = 215$ m/sec, or 770 km/hr. Because of their wavelength, their period may be as high as $T = \lambda / V_P$ = 15 to 75 minutes. In the deep ocean tsunamis may have a wave height of less than 1 meter which makes it difficult to detect passage of a tsunami in open water. On approaching the shore the wave velocity, determined by $\sqrt{gd}$, rapidly slows down to about 60 km/hr at depths of 30 meters causing the height of the wave to increase rapidly. Tsunami wave heights over 100 meters have been recorded on reaching land. This speed is still great compared to the speed at which a human can run and the tsunamis come ashore as a breaking surge of a huge wave with the capability of great damage.

55.3 Wave Interference

Another phenomena often observed at points where waves moving in directions perpendicular to one another cross paths. In the region of intersection, the principle of superposition determines the resulting wave pattern. In water, the effect observed is the sum of the two waves as illustrated in figure 9.11. Taking the direction of one wave to be along the x-axis and the other along the y-axis, the waveform in the region of interaction for two waves of equal amplitude will be

$$y = A\sin(k_1 x - \omega_1 t) + A\sin(k_2 x - \omega_2 t), \tag{55.30}$$

where the subscripts, 1 and 2, designate the parameters of each wave. Assuming that the restoring force is linear, there will be no residual effects of the interference outside the interference region. This effect may be observed when two channels open into a lake. In the open sea, the effect is frequently observed when waves from two storms intersect. Sailors call this effect "chop".

Problems

213. For deep water waves of wavelength $\lambda > 0.1\ m$ where the wave velocity is approximated by $V_P \simeq \sqrt{\frac{g\lambda}{2\pi}}$, show than an equivalent expression for the wave velocity is $V_P \simeq \frac{g}{2\pi f}$.

214. If a stone is dropped into a pool of water, the waves will radiate outward in

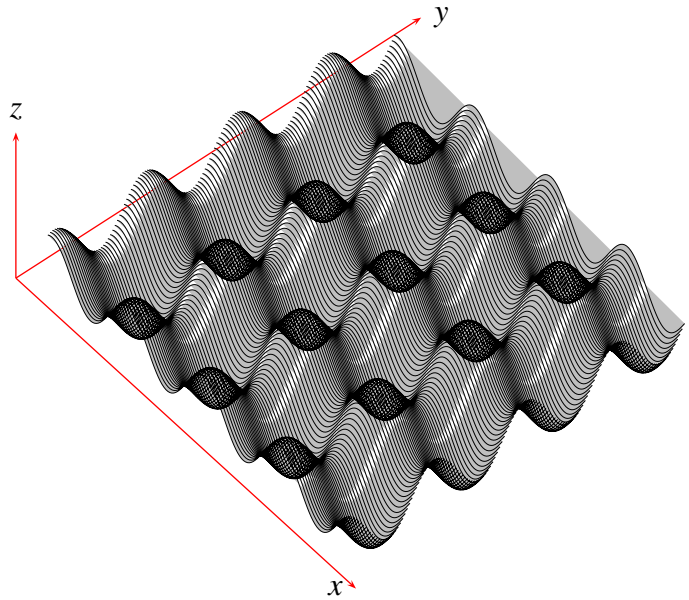


Figure 9.11: Interference pattern for two wavefronts moving perpendicularly.

a circular pattern. How is the amplitude at a distance r from the point where the stone hits the water related to the amplitude at $r = 0$?. ans. $A(r) \propto A_o/r$

215. Suppose that in deep water, an observer measures the period of a piece of driftwood bobbing up and down in the ocean current to be 10 seconds. Estimate the wave velocity, wavelength and group velocity of the waves from this one observation. ans. $V_P = 15.6$ m/sec, $\lambda = 156$ meters and $V_g = 7.8$ m/sec.

216. If the observer measures the amplitude of the waves by hanging a line overboard and noting the distance the waves rise up the line to be 2 meters, what is the energy density and intensity of the waves. ans. $E_d = 789\ Joules/m^3$ and $I = 12,304\ watts/m^2$

217. What will the wave velocity of the ocean waves in the problem above be when the water is 1 meter deep as the waves reach the shoreline? Would these waves be considered shallow water waves as they reach shore at 1 meter depth? What will their wavelength be? At what depth will the waves first begin to break? ans. 3.13 m/sec, yes, 31.3 meters, 2.6 meters.

218. If the surface tension of water were doubled, what would be the effect on wave velocity and wave length of water waves at the wind velocity where waves are first formed. ans. $V_P = 38.8$ cm/sec and $\lambda = 2.42$ cm.

56 Sound Waves

In the preceding sections we discussed transverse vibrations in a string to establish the basic wave parameters and relationships between them and surface waves on water vibrating transversely to the direction of motion. In this section, we will discuss pressure waves that vibrate parallel to the direction of motion. Pressure waves are mechanical waves, also known as longitudinal waves, and sound is a pressure or longitudinal wave. Longitudinal waves may be propagated in solids, liquids and gases and are characterized by alternate condensation or compression and rarefactions of the molecules of the medium in which the wave is transmitted. Pressure waves may be generated at any frequency. At very low frequencies, pressure waves are generated by seismic events such as earthquakes and are sometimes called **infrasonic** waves. Between the range of about 20 cycles per second and 20,000 cycles per second pressure waves are known as **audible** waves since they can be detected by the human ear. Above this frequency range, pressure waves are known as **ultrasonic** waves. An example is elastic vibrations in a quartz crystal induced by the piezoelectric effect.

When we restrict our discussion to propagation of pressure waves in one direction such as through a long metal rod or a column of water or air, the general wave equation for propagation of pressure waves in deformable material media can be written as

$$\frac{\partial^2 y}{\partial t^2} = V^2 \frac{\partial^2 y}{\partial x^2} \quad \text{where} \tag{56.1a}$$

$$V = \sqrt{\frac{dp}{d\rho}}. \tag{56.1b}$$

The coefficient V has units of velocity and reduces to certain specific forms for media of different types.

56.1 Velocity in air

In the case of sound waves in air, we can assume that the compression and rarefaction are adiabatic and use the adiabatic rule, PV^{γ}=Constant, and differentiate to obtain an expression for dp.

$$V^{\gamma} dP + \gamma P V^{\gamma-1} dV = 0 \tag{56.2}$$

$$dP = -\gamma P \frac{dV}{V} \tag{56.3}$$

Since we also presume no mass flow, we can write $m = \rho V$ and differentiate to obtain a relation between dV/V and $d\rho/\rho$.

$$dm = \rho dV + V d\rho = 0 \tag{56.4}$$

$$\frac{dV}{V} = -\frac{d\rho}{\rho} \tag{56.5}$$

Combining these two expressions we obtain the wave velocity of sound in air.

$$V_P = \sqrt{\frac{\gamma P}{\rho}} \tag{56.6}$$

Substituting $P = 1.013 \times 10^5\ N/M^2$ for atmospheric pressure at 1 atm, $\rho = 1.205$ kg/M^3 at 20 degrees Centigrade and $\gamma = 1.4$ gives the speed of sound to be 343 M/sec. It is well know that the velocity of sound in air increases with the temperature. To explain this we can use the general gas law $PV = nRT$ with $n = m/M_o$ and $\rho = m/V$ to obtain $\rho = M_o P/RT$, where M_o is the molecular weight and T is the temperature in degrees Kelvin. Using this expression for the density the wave velocity can be expressed in terms of Temperature.

$$V_P = \sqrt{\frac{\gamma RT}{M_o}} \tag{56.7}$$

Substituting $R = 8.31\ joules/mole/K$ and $T = 273.15$ degrees Kelvin gives $V_P =$ 331 M/sec at 0 degrees Centigrade. Expressing $T = 273.15(1 + T[^oC]/273.15)$ and taking T to be 20 degrees Centigrade gives a value for V_P of 343 M/sec in agreement with the previous calculation.

$$V_P = 331\sqrt{(1 + \frac{T[^oC]}{273.15})} \tag{56.8}$$

56.2 Velocity in solids

In case the material is a solid, a change in pressure, dp, produced by the compaction can be written in terms of Young's modulus, $dp = Y\frac{dx}{x}$ where Y is Young's modulus and dx is the change in the x-coordinate due to compaction. Also, the change in the density, $d\rho$, produced by the compaction can be obtained by writing the density as $\rho = \frac{m}{V}$, where $V = Ax$ is the volume containing a mass m of the solid. Then differentiating and noting that $dm = 0$, we get $d\rho = \rho\frac{dx}{x}$ where we

have taken the change as positive to correspond with an increase in pressure. Now inserting these results into equation 56.1a we obtain for the velocity of compression waves in a solid.

$$V_P = \sqrt{\frac{Y}{\rho}} \tag{56.9}$$

As an example, consider a sound wave propagating through a block of Aluminum with $Y = 6.9 \times 10^{10}\ N/M^2$ and density $\rho = 2.7 \times 10^3\ kg/M^3$. In this case the wave velocity will be $5,055\ M/second$, much larger than the velocity of sound in air. Pressure waves propagating through a solid can also vibrate transversely to the direction of propagation. In this case Young's modulus must be replaced by the shear modulus S. For Aluminum the shear modulus is $2.4 \times 10^{10}\ N/M^2$ so that the velocity of propagation for the transverse waves will be much less at $2,981\ M/second$.

56.3 Velocity in liquids

In case the material is a liquid, the change in pressure, dp, produced by the compaction can be written in terms of the Bulk modulus, $dp = B\frac{dx}{x}$ where B is the bulk modulus. The other steps in the analysis are essentially the same so that the wave velocity for a sound wave propagating through a liquid is

$$V_P = \sqrt{\frac{B}{\rho}}. \tag{56.10}$$

For example, water has a bulk modulus of about $2.2 \times 10^9\ N/M^2$ and a density of $10^3\ kg/M^3$ so that the velocity of a pressure wave propagating through water will be about 1,500 $M/second$, between that of air and solids.

The difference is wave velocities of sound passing thorough gases, liquids and solids can be understood in terms of how close the molecules of the substance are to one another. The closer the molecules are, the faster impulses can be transmitted through collisions.

56.4 Seismic waves

One specific case of interest is seismic waves, which are of four types. (1) **Primary** waves, designated as P-waves, which are longitudinal waves that travel outward from the source of an earthquake at velocities 14,000 M/sec. (2) **Secondary**

waves, designated as S-waves, which are transverse shear waves that also travel outward from the source of an earthquake but at lower velocities around 3,500 M/sec. (3) **Love** waves which are torsional surface waves which produce momentary distortions of the earth's surface. (4) **Rayleigh** waves which are also surface waves but which are transverse waves and behave like water waves. P-waves travel faster than secondary waves. Since they are longitudinal, they pass through rock, soil and water moving rock particles back and forth along the line of propagation and can penetrate through the earth typically arriving at the surface with an abrupt shock. S-waves travel slower than p-waves and move rock particles transversely to the direction of motion typically causing the first roll associated with an earthquake. P- and S-waves produce surface waves when they arrive at the surface. Surface waves do the most damage to man made structures by rocking them back and forth like a boat on an ocean wave. S-waves cannot pass through water or gas since fluids cannot support shear or transverse waves. S-waves generally do more damage than P-waves. Love waves are generated when P- and S-waves reach the surface directly above the origin of the earthquake and travel outward at velocities between 1,000 and 8,000 M/sec with periods ranging from 1 to 1000 seconds. These waves are dispersive and vibrate the ground in horizontal directions. Rayleigh waves[8] are transverse waves that roll along like water waves with wave velocities from 1,000 to 5,000 M/secand are usually responsible for most of the damage done by earthquakes.

56.5 Intensity of Sound Waves

We will start with an expression for the **pressure of sound waves**, or for that matter longitudinal waves in general, To find such an expression, we start with a sound wave passing through a long straight tube encasing the medium through which the sound wave is transmitted. The medium in the tube will consist of a series of dense and rarified regions through which the sound wave is moving. The sound wave can be described by

$$s = s_o \sin(kx - \omega t) \tag{56.11}$$

In this formula, s is the longitudinal displacement of the volume element $dV = Ads$ similar to representing the transverse displacement of volume elements in a string by y in equation 53.1, not to be confused with the coordinate x representing the position of the traveling wave. The pressure that must be exerted on a volume

[8]Existence predicted by Lord Rayleigh in 1885

$V = Adx$ to produce a fractional compression $\Delta V/V$ can be written in terms of the bulk modulus B of the material media using equation 56.11(c).

$$dp = -B\frac{dV}{V} = -B\frac{ds}{dx} \tag{56.12}$$

Differentiating equation 56.11 and making use of equation 56.10, $B = \rho V P_P^2$,this equation can be modified to read eq:soundpressure2

$$dp = po\cos(kx - \omega t) \quad \text{where} \tag{56.13a}$$

$$p_o = \rho V_P^2 k s_o \quad \text{or} \tag{56.13b}$$

$$p_o = \rho V_P \omega s_o \quad \text{Pa} \tag{56.13c}$$

We know that the maximum pressure that the human ear can withstand before pain sets in is about 28 Pa (equal to about 0.04 psi) at 1,000 Hz. This reference information can be used to place the above parameters into perspective. Using $p_o = 28$ Pa and equation **??**(c) we find that this pressure level corresponds to a displacement level s_o of only about 1×10^{-5} meters, or 10 *micrometers*. It may also be noted that the wavelength of sound in the mid range of human hearing at 1000 Hz is $\lambda = 331/1000$ = 34 cm. Thus, very loud sounds arise from very small displacements of air.

The intensity of a sound wave is defined as the rate at which energy crosses a unit area. The energy being transmitted consists of both kinetic and potential energy. The kinetic energy dK contained in a mass element $dm = \rho A dx$ is obtained from

$$K = \frac{1}{2} dm V_s^2 \tag{56.14a}$$

$$= \frac{1}{2}\rho A dx \omega^2 s_o^2 \cos^2(kx - \omega t) \quad \text{where} \tag{56.14b}$$

The average kinetic energy is obtained by noting that the average value of the cosine squared term integrated over one wavelength is 1/2 so that

$$\bar{K} = \frac{1}{4}\rho A dx \omega^2 s_o^2 \tag{56.15}$$

From the equipartition theorem we know that the average kinetic energy equals the average potential energy so that the average total energy $\bar{E} = 2\bar{K}$ and the rate at which energy passes a unit area, or the Intensity, is

$$I = \frac{1}{A}\frac{d\bar{E}}{dt} = \frac{1}{2}\rho V_P \omega^2 s_o^2 \quad watts/m^2 \tag{56.16}$$

The intensity of the sound wave has the same units as power radiated per unit area. In the above explanation it is assumed that all the power is transmitted down the tube encasing the medium. However, it is often convenient to think in terms of sound being generated at a source buried inside a large medium with sound waves being emitted in all directions. In this case the appropriate wave equation must be written in spherical coordinates.

$$\frac{\partial^2 y}{\partial t^2} = V^2 \frac{\partial^2 y}{\partial x^2} \tag{56.17}$$

The same definitions apply as for the one dimensional case, but the solution of the spherical wave equation is properly written as

$$\phi = \frac{A}{r} \sin\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t\right) \tag{56.18}$$

The intensity of the pressure wave is then obtained in the same way and found to be

$$I = \frac{1}{2} \rho V \omega^2 \frac{A^2}{r^2} \tag{56.19}$$

and decreases as the inverse square of the distance from the source as opposed to the intensity of a plane waves discussed in section 53 which is constant. To make it easier to perform calculations in spherical emissions, we define normally define the intensity of the sound wave in terms of the power of the source emitting the waves.

$$I = \frac{P}{4\pi r^2} \tag{56.20}$$

In this formula, P is the power emitted by the source in watts and the intensity is, as before in watts/square meter.

The human ear responds to sound with signals to the brain that are proportional to the logarithm of the amplitude of vibration of the basil membrane instead of the amplitude itself. This makes it possible for the human ear to process signals that vary from very low intensity to very high intensity. It also makes it desirable to have a physiological unit to measure the sound level that incorporates the logarithm of the intensity.

$$L = 10 \log\left(\frac{I}{I_o}\right) \tag{56.21}$$

On this scale, the constant I_o is taken to be $1 \times 10^{-12}\ watts/M^2$ and corresponds to the threshold of hearing in normal human ears. The loudness L is a unitless term

expressed **decibels**. The level of pain on this scale is about 120 decibels, which corresponds to an intensity of about 1 $watt/m^2$, a scale factor of about 10^{20}. The average intensity of normal speech is about 7×10^{-6} $Watts/cm^2$ at a point 9 cm from the mouth.[9]

56.6 Doppler shift

The change in pitch of a train whistle or car horn as it passes a stationary observer is well known. This effect was first predicted theoretically in 1842 by Johann Doppler (1803-1853) an Austrian mathematician and physicist and experimentally observed in 1845 by Buys Ballot (1817-1890) a Dutch chemist and metrologist. If a source is emitting waves of frequency f_o and wavelength λ_o that move with a wave velocity V_o a stationary observer will observe the wave to have these parameters. If the source begins to move relative to the stationary observer at a constant speed of V_s the stationary observer will observe the passage of wave fronts more frequently and having no way to know the source is moving will believe the frequency of the waves being emitted by the source is

$$f_{obs} = f_o \frac{V_o}{V_o \pm V_s}. \tag{56.22}$$

where the - sign applies if the source is moving toward the receiver and the + sign if the source is moving away. As a result, the observer will believe the frequency has increased if the source is moving toward him and decreased if the source is moving away from him. Similarly, if the observer is moving relative to the source, he will observe a different rate of passage of the wave fronts and believe the frequency of the waves being emitted by the source is

$$f_{obs} = f_o \frac{V_o \pm V_r}{V_o}. \tag{56.23}$$

where the + sign applies if the receiver is moving toward the source and the - sign if the receiver is moving away. This effect, known as the Doppler shift, appears in all observations of wave motion, whether it is listening to the passage of a train blowing its horn as it passes an observer or observing light from a distant star. These formulas can be combined to obtain.

$$f_{obs} = f_o \frac{V_o \pm V_r}{V_o \mp V_s}. \tag{56.24}$$

where the numerator uses the sign conventions apply as before.

[9] 1 $watt/m^2$ = 100 $microwatts/cm^2$.

56.7 Audible Range

The pressure waves detected by the human ear, commonly referred to as sound, are described in table 9.2. Other species such as bats and dolphins also use sound waves for communication and navigational purposes. The wide range of the response of human ears is a testimony to perfection. The frequency range of sound usable in bats and other species may range up to 100,000 Hertz above the range usable for humans. The speed of sound in air at standard temperature and pressure is 343 Meters/second so that the product of frequency and wavelength must be $f\lambda = 344$. As an example, the 49th key of a modern piano is normally tuned to 440 Hz so that the wavelength of the note heard is 0.78 meters or about 26 inches.

Parameter	Minimum	Maximum	Units
Frequency	20	20,000	Hertz
Pressure	10^{-12}	10	$Watts/meter^2$
	0	130	decibels
	2×10^{-5}	60	Pascals
	2×10^{-10}	6×10^{-4}	Atmospheres
Wavelength	0.0172	17.2	Meters
	0.0567	56.4	Feet

Table 9.2: Audible Range

56.8 Harmonics

Just as harmonics are developed in a vibrating string, harmonics are developed as a result of pressure waves traveling along a tube. The wave velocity, wave length and frequency are related by $V_P = f\lambda$ as for strings. In this case, however, the speed of sound is constant at about 343 $meters/second$ and the wavelength is determined by the length of the tube. Pressure waves traveling along a tube are reflected at the end of the tube whether the tube is closed or not. If the tube is closed at the end, the reflected wave is 180 degrees out of phase with the incoming wave and the end of the tube is a displacement node. If the tube is open at the end and narrow compared to the wavelength of the traveling wave, the reflected wave is in phase with the incoming wave and the end of the tube is a displacement antinode. The natural frequencies of vibration for a tube are calculated using the

λ	frequency	Overtone	Harmonic
$\frac{2L}{1}$	$\frac{V}{2L}$	Fundamental	First harmonic
$\frac{2L}{2}$	$\frac{2V}{2L}$	First Overtone	Second harmonic
$\frac{2L}{3}$	$\frac{3V}{2L}$	Second Overtone	Third harmonic
$\frac{2L}{4}$	$\frac{4V}{2L}$	Third Overtone	Fourth harmonic
$\frac{2L}{5}$	$\frac{5V}{2L}$	Fourth Overtone	Fifth harmonic
$\frac{2L}{6}$	$\frac{6V}{2L}$	Fifth Overtone	Sixth harmonic

Table 9.3: Harmonic series for open organ pipes.

λ	frequency	Overtone	Harmonic
$\frac{4L}{1}$	$\frac{V}{4L}$	Fundamental	First harmonic
$\frac{4L}{3}$	$\frac{3V}{4L}$	First Overtone	Second harmonic
$\frac{4L}{5}$	$\frac{5V}{4L}$	Second Overtone	Third harmonic
$\frac{4L}{7}$	$\frac{7V}{4L}$	Third Overtone	Fourth harmonic
$\frac{4L}{9}$	$\frac{9V}{4L}$	Fourth Overtone	Fifth harmonic
$\frac{4L}{11}$	$\frac{11V}{4L}$	Fifth Overtone	Sixth harmonic

Table 9.4: Harmonic series for open organ pipes.

same formulas as for a vibrating string.

$$f_n = \frac{n}{2L} V_P \tag{56.25}$$

The harmonic series for an open organ pipe with both ends open is presented in table 9.3. The fundamental frequency corresponds to a displacement antinode at both ends and a displacement antinode in the middle of the pipe. This results in a wavelength of 2L and therefore a fundamental frequency of $V/2L$.

The harmonic series of a closed organ pipe is presented in table 9.4. In this case, the closed end is a displacement node and the open end is a displacement antinode resulting in a fundamental wave length of 4L and therefore a fundamental frequency of $V/4L$.

Problems

219. What is the speed of sound in air at 100 degrees C? ans. 387 m/sec

220. What is the speed of sound in structural steel? Take $Y = 2.00 \times 10^{11}\ N/m^2$ and $\rho = 7850\ kg/m^3$. ans. 5048 m/sec

221. If sound passes through 100 meters of structural steel beam what will be the time delay between arrival of the sound at the end of the beam and arrival of the same signal passing through air beside the steel beam? ans. 0.272 seconds

222. What is the speed of sound in SAE 30 motor oil for which $Y = 1.3$ GPa and $\rho = 737\ kg/m^3$? ans. 1328 m/sec

223. Given that the velocity of longitudinal P-waves is 14,000 m/sec, the velocity of transverse S-waves is 3,500 m/sec and that the velocity of shear Love waves is 3,000 m/sec and transverse Rayleigh waves is 4,000 m/sec that originate at the surface above an earthquake at a certain area in the earth's crust, calculate the time of arrival of these waves at a point 100 km from the epicenter of an earthquake which occurs 5,000 m below the surface. ans. 7.15 sec, 28.57 sec, 25.03 sec and 33.37 sec.

224. A speaker emitts power at 200 watts from the top of a tower 50 m tall. What is the intensity of the sound at a point on the ground a distance of 50 m from the base of the tower and what is the decibel level of the sound? ans. 3.18 milliwatt, 95 dB

225. What pressure will a sound wave of intensity 1 $watt/m^2$ exert on a surface? ans. 0.89 Pa

226. What is the amplitude of oscillations in a longitudinal a 1000 Hz sound wave of 1 $watt/m^2$ intensity? ans. 0.035 cm

227. You are approaching a police car at 75 mph in a 50 mph zone and the police car sounds a siren at 1000 Hz. What frequency do you hear? ans. 1101 Hz

228. You ignore the siren and drive away at 75 mph. What frequency do you hear before the police car gives chase? 899 Hz

229. The police car gives chase at 114 mph. Then what frequency do you hear? ans. 1062.5 Hz

57 Light rays and Electromagnetic radiation

57.1 Maxwells Equations

We digressed at four different points in this text to derive Maxwell's equations relating the electromagnetic vectors $\vec{\mathbf{E}}$, $\vec{\mathbf{D}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$. The results were as follows:

$$\nabla \cdot \vec{\mathbf{D}} = \rho \tag{57.1a}$$

$$\nabla \times \vec{\mathbf{H}} = J + \frac{\partial \vec{D}}{\partial t} \tag{57.1b}$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{B}}{\partial t} \tag{57.1c}$$

$$\nabla \cdot \vec{B} = 0 \tag{57.1d}$$

In free space, the charge density ρ and current density $\vec{\mathbf{J}}$ are zero. Each of these field vectors have a wave equation that describes their variation in space and time. For example, we can obtain the wave equation satisfied by $\vec{\mathbf{E}}$ by eliminating $\vec{\mathbf{H}}$ from equation 57.1a(b). To do this we simply take the curl of both sides of equation 57.1a(b) and make use of $\vec{\mathbf{D}} = \varepsilon_o \vec{\mathbf{E}}$.

$$\nabla \times \left(\nabla \times \vec{\mathbf{E}}\right) = -\mu_o \frac{\partial}{\partial t} \nabla \times \vec{\mathbf{H}} \tag{57.2}$$

Substituting into equation 57.1a(c) we obtain a wave equation satisfied by $\vec{\mathbf{E}}$.[10]

$$\nabla^2 \vec{\mathbf{E}} = \varepsilon_o \mu_o \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{57.3}$$

Similar derivations may be performed to establish wave equations for $\vec{\mathbf{H}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{D}}$. As may be verified by substitution, equation 57.3 is solved by a wave function of the type.

$$E = E_o \cos\left(kx - \omega t\right) \tag{57.4}$$

where $k = 2\pi/\lambda$ and $\omega = 2\pi/T$. Substituting into equation 57.3 and simplifying, we obtain the dispersion relation for the electric vector.

$$k^2 = \varepsilon_o \mu_o \omega^2 \tag{57.5}$$

[10] Making use of the vector identity $\nabla \times \left(\nabla \times \vec{\mathbf{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathbf{E}}\right) - \nabla^2 \vec{\mathbf{E}}$.

This allows us to calculate the wave velocity.

$$c = \frac{1}{\varepsilon_o \mu_o} \tag{57.6}$$

The constants ε_o and μ_o can be measured experimentally and their measured values used to calculate the wave velocity of electromagnetic waves.

$$\varepsilon_o = 8.854187817 \times 10^{-12} \quad \text{Farads/meter} \tag{57.7}$$

$$\mu_o = 4\pi \times 10^{-7} \quad \text{Henry/meter} \tag{57.8}$$

$$c = 2.9979250 \times 10^{8} \quad \text{meters/second} \tag{57.9}$$

The theory of electromagnetic waves was formulated in 1864 by James Clerk Maxwell (1831-1879) a Scottish theoretical physicist and mathematician. His calculated value for the wave velocity of electromagnetic waves was confirmed in 1889 by the German physicist Heinrich Rudolf Hertz (1857-1894) who was professor of physics at the Polytechnic Institute, Karlsruhe, Germany.

57.2 Propagation of Electromagnetic radiation through space

From equation 57.1a(c) it is seen that the curl of the vector $\vec{\mathbf{E}}$ produces the time derivative of the vector $\vec{\mathbf{H}}$. Since the curl of a vector is at right angles to the vector, it is then clear that the vectors $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ oscillate in perpendicular planes. The propagation of an electromagnetic wave through space is illustrated in figure 9.12 which shows the electric field vector as the solid line oscillating in the yz-plane and the magnetic field vector oscillating in the xy-plane.

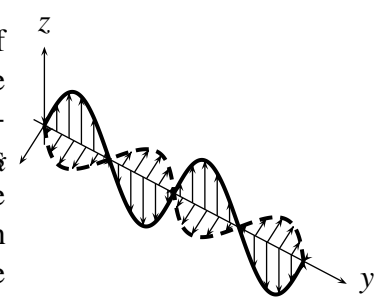


Figure 9.12: E and H vectors in electromagnetic wave.

If we take the direction of propagation as lying along the x-axis so that we can take the plane of oscillation of the electric vector as the xz-plane and the plane of oscillation of the magnetic vector as the xy-plane, the wave equations for the electric vector and its solution can be written

$$\frac{\partial^2 E_z}{\partial x^2} = \varepsilon_o \mu_o \frac{\partial^2 E_z}{\partial t^2} \tag{57.10}$$

$$E_z = E_{zo} \sin(kx - \omega t) \tag{57.11}$$

while the wave equation for the magnetic vector and its solution can be written

$$\frac{\partial^2 H_z}{\partial x^2} = \varepsilon_o \mu_o \frac{\partial^2 H_z}{\partial t^2} \tag{57.12}$$

$$H_y = H_{yo} \sin(kx - \omega t) \tag{57.13}$$

From this it is apparent that the electric and magnetic fields of an electromagnetic wave propagate in the same direction, are mutually perpendicular and oscillate in phase with one another. It is also seen by inserting the solution for $\vec{\mathbf{H}}$ into Maxwell's second equation that the ratio of the magnitudes of the electric and magnetic vectors equals the velocity of the wave.

$$\nabla \times \vec{H}_o \sin(kx - \omega t) = \varepsilon_o \frac{\partial}{\partial t} \vec{E}_o \sin(kx - \omega t) \tag{57.14}$$

$$\sin kx - \omega t) \nabla \times \vec{H}_o + \vec{H}_o \times \nabla \sin(kx - \omega t) = -\varepsilon_o \omega \vec{E}_o \cos(kx - \omega t) \tag{57.15}$$

$$\vec{H}_o \times \hat{\mathbf{i}} k \cos(kx - \omega t) = -\varepsilon_o \omega \vec{E}_o \cos(kx - \omega t) \tag{57.16}$$

$$k\vec{H}_o = \varepsilon_o \omega \vec{E}_o \tag{57.17}$$

where we have noted that $\nabla \times \vec{H}_o = 0$ since $\vec{H}_o$ is a constant vector. Substituting $c = \omega/k$ we find that

$$\frac{E_o}{H_o} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 376.7 ohms \tag{57.18}$$

which is considered the "impedance" of free space. Making use of $B = \mu H$, we also obtain

$$\frac{E_o}{B_o} = c \tag{57.19}$$

57.3 Poynting vector

We have already seen that the energy density of an electric field is $\frac{1}{2}\varepsilon_o E^2$ and that of a magnetic field is $\frac{1}{2}\mu_o H^2$. Therefore the total **energy density** of the electromagnetic field is

$$u = \frac{1}{2}\left(\varepsilon_o E^2 + \mu_o H^2\right) \tag{57.20}$$

Taking the cross product of the vectors $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$, we immediately see that[11]

$$\nabla \cdot \left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}\right) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_o E^2 + \mu_o H^2\right) \tag{57.21}$$

[11] Making use of the identity $\nabla \cdot \left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}\right)$ = - $\mu_o \vec{\mathbf{E}} \cdot \frac{\partial \vec{\mathbf{E}}}{\partial t}$ - $\varepsilon_o \vec{\mathbf{H}} \cdot \frac{\partial \vec{\mathbf{H}}}{\partial t}$

which represents the rate of decrease of the electromagnetic energy per unit volume of the field. Now, integrating to obtain the rate of decrease of the total electromagnetic energy in a given volume and converting to a surface integral using the divergence theorem we obtain[12]

$$\int_V \nabla \cdot \left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}\right) dV = \int_S \left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}\right) \cdot \hat{\mathbf{n}} dS = -\frac{1}{2}\frac{\partial}{\partial t}\int_V \left(\varepsilon_o E^2 + \mu_o H^2\right) dV. \quad (57.22)$$

which gives the rate at which energy flows out of the volume V across a surface S. This prompts us to define the **Poynting** vector as

$$\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}}. \quad (57.23)$$

This vector, named after the British physicist J. A. Poynting (1852-1914) who first discovered its properties, is perpendicular to the directions of the electric and magnetic vectors, points in the direction of propagation and has units of $watts/m^2$. The Poynting vector $\vec{\mathbf{S}}$ represents the flux of energy through the surface of a volume. Therefore, the total surface integral of $\vec{\mathbf{S}}$ contributes to the total electromagnetic energy balance in a volume enclosed by the surface. Use of the Poynting vector to identify energy flow at any one point may lead to inconsistencies.[13]

$$Power = \oint_S \vec{\mathbf{S}} \cdot \hat{\mathbf{n}} dS \quad (57.24)$$

57.4 Intensity of EM wave

As a matter of semantics it may be recalled that intensity or power in $watts/m^2$ is the product of energy density $joules/m^3$ and wave velocity m/sec. This equivalence holds for instantaneous intensity as well as average intensity provided that instantaneous energy density or average energy density is used in the appropriate case. Also it should be noted that the instantaneous and average energy density of an electromagnetic wave are

$$u = \frac{1}{2}\left(\varepsilon_o E^2 + \mu_o H^2\right) \quad (57.25a)$$

$$\bar{u} = \frac{1}{4}\left(\varepsilon_o E^2 + \mu_o H^2\right) \quad (57.25b)$$

[12] $\int_V \nabla \cdot \vec{\mathbf{F}} dV = \int_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} dS$.

[13] W.K.H Panofsky and M. Phillips, "Classical Electricity and Magnetism" Addisonj-Wesley Publishing Company, Inc., Reading, Massachusetts (1955) p. 161

where we have calculated the average value $\overline{\sin^2 kx} = \frac{1}{2}\sin^2 kx$. This allows us to calculate the instantaneous and average intensity or power of the electromagnetic wave.

$$I = EH = \sqrt{\frac{\varepsilon_o}{\mu_o}}E^2 \tag{57.26a}$$

$$\bar{I}_{avg} = \frac{\overline{S}}{2} = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o^2 \tag{57.26b}$$

where we have used equation 57.19. It should also be noted that if the electromagnetic waves spread out omnidirectionally from a point in space, the intensity will decrease as the area of the sphere increases.

$$I = \frac{I_o}{4\pi r^2} \tag{57.27}$$

where I_o is the intensity or power of the emitted radiation and r is the distance from the source.

57.5 Momentum carried by EM wave

Based on dimensional analysis, it is easy to see that the **momentum** carried by an electromagnetic wave is energy divided by the wave velocity; therefore the average momentum per unit volume carried by an electromagnetic wave is

$$\bar{p} = \frac{1}{c^2}\bar{S}. \tag{57.28}$$

This is the momentum imparted to an object absorbing all the radiation incident upon it. If all the radiation was reflected, the momentum imparted to the object would be twice the amount in the above formula.

57.6 Radiation pressure of EM wave

Also based on dimensional analysis, the intensity divided by the wave velocity is a force per unit area, or a pressure. Therefore, we can define the average **radiation pressure** of the electromagnetic wave as the average intensity divided by c.

$$\bar{P}_{rad} = \frac{1}{2c}\bar{S} \tag{57.29}$$

Example

Consider a light bulb emitting radiation at the rate of 1000 watts and we wish to know as much as possible about the electromagnetic waves at a distance of 10 meter from the light source. We can start by calculating the intensity of the radiation at 1 meter from the source.

$$I = \frac{1000}{4\pi(10)^2} = 0.796 \ \ watts/m^2 \tag{57.30}$$

The maximum amplitude of the electric power vector at 1 meter from the source is

$$E_o = \sqrt{2I\sqrt{\frac{\mu_o}{\varepsilon_o}}} = \sqrt{(2)(0.796)\sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}}} = 24.5 \, Volts/meter \tag{57.31}$$

The maximum amplitude of the magnetic vector is then

$$B_o = \frac{E_o}{c} = \frac{24.5}{3 \times 10^8} = 8.17 \times 10^{-8} \, Teslas \tag{57.32}$$

The radiation pressure of the electromagnetic wave is

$$\overline{P}_{rad} = \frac{I}{c} = \frac{0.796}{3 \times 10^8} = 2.65 \times 10^{-7} \, Pascals \tag{57.33}$$

and the momentum carried by the electromagnetic wave is

$$p = \frac{\overline{S}}{c^2} = 1.76 \times 10^{-17} N - sec \tag{57.34}$$

57.7 Polarization

In the study of electrodynamics three types of polarization of an electromagnetic wave have been defined: (1) linear, (2) circular and (3) elliptical. In principle, polarization may refer to polarization of either the electric or magnetic vector; but historically polarization has referred to the electric vector in the optical band and to the magnetic vector in the radio band.

Linear polarization occurs when either the electric or magnetic vector is confined to oscillate in one plane along the direction of propagation.

Circular polarization If light is composed of two plane waves of equal amplitude but differing in phase by 90âĹđ, then the light is said to be circularly polarized. If you were looking at the source of the radiation and could see the tip of the electric field vector, it would appear to be moving in a circle as it approaches you.

Elliptical polarization If light is composed of two plane waves of arbitrary amplitude and arbitrary phase, the light is said to be elliptically polarized. If you were looking at the source of the radiation and could see the tip of the electric field vector, it would appear to be moving in an ellipse as it approaches you.

If the tips of the vectors appear to rotate in a clockwise fashion about the direction of propagation, the wave is said to have **positive helicity** and if in a counter-clockwise direction to have **negative helicity**.

The light that you see reflected from most surfaces or emitted by the sun, a candle, or a light bulb is **unpolarized**. The electric and magnetic vectors are not constrained to vibrate in any plane and therefore vibrate in random directions.

Polarization is a property of certain types of waves that describes the orientation of their oscillations. Electromagnetic waves, such as light, and gravitational waves exhibit polarization but pressure waves in a gas or liquid do not have polarization because the direction of vibration and direction of propagation are the same. In a solid medium, however, pressure waves can be transverse. In this case, the polarization is associated with the direction of the shear stress in the plane perpendicular to the propagation direction. In electromagnetic waves the most general form of polarization is elliptical with linear and circular being two extreme cases.

The physical orientation of a wireless antenna corresponds to the polarization of the radio waves received or transmitted by that antenna. Thus, a vertical antenna receives and emits vertically polarized waves, and a horizontal antenna receives or emits horizontally polarized waves. The best short-range communications is obtained when the transmitting and receiving antennas have the same polarization. The least efficient short-range communications usually takes place when the two antennas are at right angles, for example, one horizontal and one vertical. This same principle can be used to shield against microwave radiation. For example, if microwaves are emitted by an antenna composed of horizontal wires a screen of horizontal wires placed in front of the antenna will absorb the radiation while a screen with vertical wires placed in front of the antenna will transmit the radiation. The reason is that the electric vector oscillates horizontally because the antenna is horizontal so that the microwave radiation is horizontally polarized. The horizontally polarized electric vector will set up currents in horizontal conducting wires

so that energy is absorbed; however the horizontally polarized electric vector does not set up currents in the vertical wires so that little or no energy is absorbed.

There are also other sources of radiation that are polarized. For example the radiation emitted from certain atoms is found to be polarized.

There are essentially three ways to polarize unpolarized light including: (1) reflection, (2) refraction and (3) diffraction which will be studied in subsequent sections. The polarization of light can be measured with a polarimeter.

57.8 The Electromagnetic spectrum

The electromagnetic spectrum is illustrated in figure 9.13 broken down by wavelength. A good deal can be learned about the origin of electromagnetic waves, their effect and use in classifying the spectrum by wavelength. The bands in figure 9.13 are generally correct, but there may be considerable overlap at the boundaries.

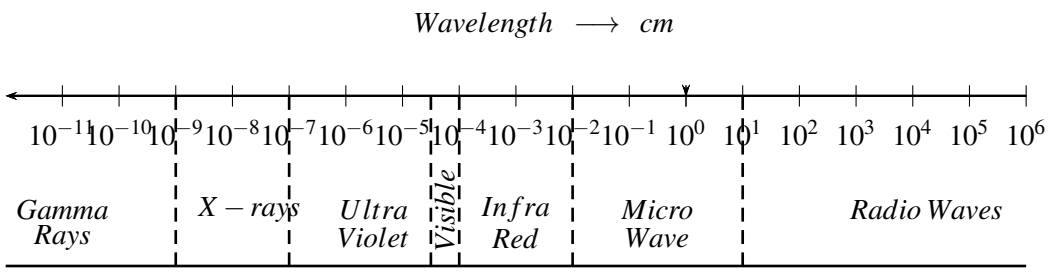


Figure 9.13: Electromagnetic spectrum

1. **radio waves** The longest wavelengths ranging from about 1 cm up to several hundred meters and are called radio waves. Radio waves are produced in antennas of comparable dimensions by oscillating currents and are used for transmission of radio and television signals and other types of data. Signal transmission is accomplished through modulation of the amplitude of the waves or the frequency. The effect of these electromagnetic waves on receivers is generally to induce currents similar to their origin. This range is divided into UHF, VHF, FM and AM bands. Radio waves normally penetrate buildings and other structures but are stopped by conducting materials since the electric field is normally reduced to zero inside conductors.

2. **Microwaves** Microwaves range from 0.01 to 10 cm in length and have energies between 0.01 and 10^{-85} eV and are produced in klystron and magnetron tubes. Because of their short wavelength, they are easily absorbed

in water making them useful in microwave ovens where they can deliver a large amount of energy in a short time.

3. **Infrared waves** Infra red waves range from about 0.7 micron up to about 0.01 cm. As noted earlier, all objects radiate energy and most of the radiated energy lies in the infra red range. Water absorbs energy strongly in this wavelength range so that moisture in the atmosphere removes a large portion of infrared radiation from the sun. However, infra red radiation penetrates many other materials which makes it useful in detecting radiating bodies because of their heat radiation. Some animals, such as pit-vipers, detect their prey by detecting its infra red radiation.

4. **Visible** The visible spectrum ranges from about 0.4 to 0.7 microns and is the means by which all animal life can see their surroundings. Visible radiation of different wavelength stimulates the eye differently so that the wavelength is recognized as a color.

5. **Ultraviolet waves** Ultraviolet waves range in wavelength from 0.4 microns down to about 0.001 microns and has both harmful and therapeutic effects on humans.

6. **X-Rays** X-Rays range in wavelength from about 1000 micromicrometers to 10 micromicrometers and have energies from 1000 to 100,000 electron volts. X-Rays are produced by bombarding high Z metals with electrons of similar energy. X-Rays deposit their energy in passing through material media by ionizing atoms along their path and may penetrate several centimeters into most materials. They are often used to image the body in medical clinics to discover broken bones and other disorders.

7. **Gamma Rays** Gamma Rays occupy the higher energy range of electromagnetic waves and are generally produced by excitation of atoms. Gamma Rays interact with material by three processes: photoelectric effect, Compton scattering and pair production which will be discussed in the next chapter.

According to all available theoretical and experimental evidence, it is the electric vector rather than the magnetic vector to which the eye is sensitive. This can be understood by noting that the magnitudes of the electric and magnetic vectors are related by the speed of electromagnetic waves, $E = cB$ and that the electric

field exerts a force on electrons of the material it penetrates while magnetic fields exert an appreciable force only if the electrons are moving.

Problems

230. The radiation power output from our sun is thought to be about 3.90×10^{26} watts. Calculate the intensity, the maximum amplitude of the electric and magnetic fields, the radiation pressure and momentum carried by the electromagnetic waves at the earth's position. ans. 1380 $watts/m^2$, 1019 v/m, 3.4×10^{-6} T, 4.6×10^{-6} Pa, 3.06×10^{-14} N-s

231. A 1-KW laser having a wavelength of 633 nm and focal point or radius about 2 times the wavelength is focused on a reflecting mirror. Calculate the intensity of the beam, the maximum amplitude of the electric and magnetic fields of the laser, the radiation pressure on the mirror and the momentum imparted to the mirror. ans. 2.00×10^{14} $watts/m^2$, 3.88×10^8 v/m, 1.29 T, 6.67×10^5 Pa, 8.88×10^{-3} N-s

232. A 3.0 milliwatt laser pointer creates a spot on a screen that is 2.0 mm in diameter. Determine the average value of the intensity of the beam, the maximum amplitude of the electric and magnetic fields of the laser, the radiation pressure on the screen and the momentum imparted to the screen which reflects 70% of the light incident on it. ans. 955 $watts/m^2$, 8.47×10^2 v/m, 2.83×10^{-6} T, 3.18×10^{-6} Pa, 3.61×10^{-14} N-s

233. The sun delivers about 1000 $watts/m^2$ in the vicinity of the earth. Calculate the average radiation pressure on a solar sail which is 1000 m^2 in area and the acceleration this pressure would impart to a 100 kg mass. ans. 3.33×10^{-6} Pa, 3.33×10^{-3} N, 3.33×10^{-5} m/sec^2

58 Reflection and Refraction of Light waves

The preceding sections were devoted to specialized waves. In this section, we will restrict our discussion to electromagnetic radiation in the visible band although results obtained in this section can be applied to waves of any type with only minor modifications to take account of the features characteristic of the wave type. This statement includes waves traveling down a string, water waves, sound or pressure waves, light waves and electromagnetic waves.

58.1 General Laws of Reflection and Refraction

In general when a ray of light strikes a transparent substance in which the velocity of light is appreciably different, it is divided into a reflected ray and a refracted ray. In figure 9.14 a light ray is reflected from the surface of a dielectric and refracted into the dielectric. The angle of incidence is θ and the plane formed by the incident ray and the normal line to the surface will be termed the plane of incidence. The angle of reflection is θ' and the plane formed by the reflected ray and the normal line to the surface will be termed the plane of reflectance. The **law of reflection** is *the reflected ray lies in the plane of incidence and the angle of reflection equals the angle of incidence*.

$$\theta = \theta' \tag{58.1}$$

Similarly, the angle of refraction is θ'' and the plane formed by the refracted ray and the normal line to the surface will be termed the plane of refractance. The **law of refraction** is that *the refracted ray lies in the plane of incidence and the ratio of sine of the angle of refraction to the sine of the angle of incidence is a constant.*

$$\frac{\sin\theta}{\sin\theta''} = \frac{n'}{n} \tag{58.2}$$

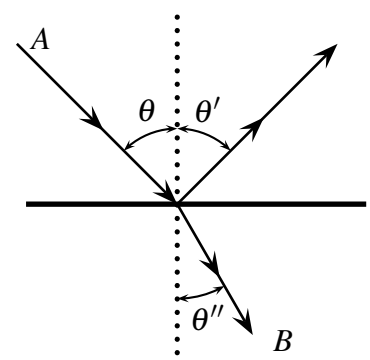


Figure 9.14: Reflection and refraction of light ray at a boundary.

The law of refraction is called **Snell's law**, and is named after Willebrord Snell (1591-1626) at the University of Leyden, Holland.

Its derivation is based on the presumption that when a light wave oscillating with a frequency f and wavelength λ_1 and traveling with the velocity $v_1 = f\lambda_1$, in medium 1 enters another transparent medium 2, its frequency, and therefore its energy, will remain the same but the wavelength will become λ_2 and the wave velocity v_2 such that $v_2 = f\lambda_2$.

58.2 Principle of Reversibility

One particularly useful observation is called the **principle of reversibility**. This principle can be stated as *If a reflected or refracted ray be reversed in direction*

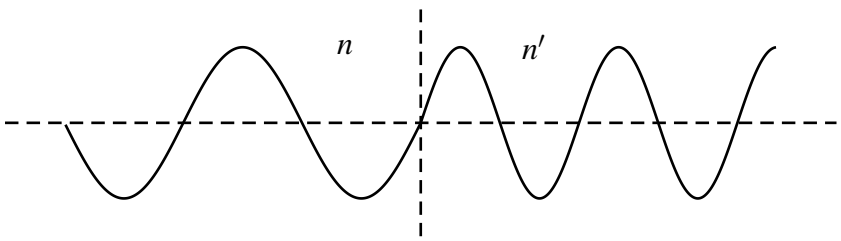


Figure 9.15: Change of wavelength and wave velocity as wave passes through boundary of two media with different indices of refraction.

it will retrace its original path. This principle shows up in the study of classical mechanics. The principle of reversibility also makes it clear that if a ray of light passing from a less dense medium to a more dense medium is bent toward the normal to the surface, that ray of light would be bent away from the normal in the reversed path passing from a medium more dense to a medium less dense. This is the case for water. A ray of light passing from air into water is bent toward the normal and a ray of light passing from water into air is bent away from the normal. This is why objects under water appear closer than they really are to an observer standing on land.

58.3 Optical Path

Another definition that will be useful in derivation of other laws of optics is the **optical path**. The optical path is defined as the product of the index of refraction and the distance traveled by the wave in that media. Letting the distance traveled by the incident ray in figure 9.14 before incidence on the boundary be represented by d and the distance after incidence by represented by d', the optical path can be defined as

$$[d] = nd + n'd' \tag{58.3}$$

Fermat's principle, named after Pierre Fermat (1608-1665) a French mathematician, states that *the path taken by a light ray in going from one point to another through any set of media is such as to render its optical path equal in the first approximation to other paths closely adjacent to the actual one.* Fermat justified this principle in his belief that "nature is economical" meaning that the optical path between two points should be a minimum.[14] Fermat's principle forms

[14]There are, however, cases in which the optical path is a maximum and cases in which it is simply constant.

a particularly simple and straightforward way to derive the laws of reflection and refraction presented above in equations 58.1 and 58.2 by first defining the formula for the optical path, differentiating and setting equal to zero. Taking the coordinates of points A and B in figure 9.14 as $(-x_1, y_1)$ and $(x_2, -y_2)$ and eliminating the dependent variables by setting $h_1 = y_1$, $h_2 = y_2$ and $x_2 = p - x_1$ where p is the total distance between points x_1 and x_2, we have for the optical path

$$[d] = n[h_1^2 + (p-x)^2]^{1/2} + n'(h_2^2 + x^2)^{1/2}. \tag{58.4}$$

Differentiating with respect to x_1 and setting equal zero gives.

$$n\left(\frac{p-x}{d}\right) = n'\left(\frac{x}{d'}\right). \tag{58.5}$$

Since the factors in parenthesis are the sines of θ and θ'' respectively, we obtain equation 58.2, Snell's law. The law of reflectance can be obtained in a similar way.

58.4 Wavefronts

Another useful way to visualize reflection and refraction of waves is through the bending of wavefronts. If we define a wavefront as an imaginary line connecting all points in space reached by a wave or vibration at the same instant as the wave travels through a medium, we can represent propagating wavefronts in the reflection and refraction of waves as illustrated in figure 9.16. This figure is drawn for an index of refraction $n = 1.4$ and an angle of incidence of 45 degrees so that the angle of reflection would also be 45 degrees and the angle of refraction would be 30 degrees. In this case the wave front AB is rotated through an angle of $45 + 45 = 90$ degrees and propagates outward with the reflected waves. However the wavefront AC is rotated through an angle of $30 - 45 = -15$ degrees and propagates into the medium with the refracted rays.

58.5 Fresnel's Law of Reflection

Finally, we consider **Fresnel's law of reflection** which is very useful in explaining observed phenomena as was Fermat's principle in deriving Snell's law from theoretical considerations. Fresnel's law is named for a French Engineer, Augustin Fresnel (1788-1827) who discovered the law. Fresnel's law can be derived from electromagnetic wave equations by requiring continuity of the electric vector at

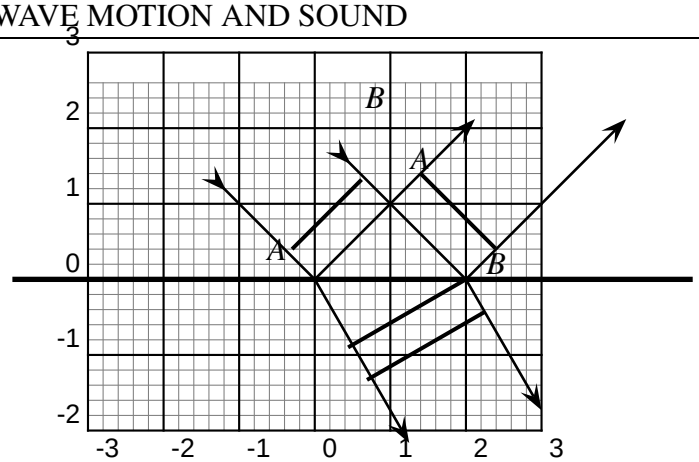


Figure 9.16: Reflection and refraction of light ray at a boundary.

the interface between the two media. Returning to figure 9.14, we can define the amplitude of the electric vector in the incident, reflected and refracted light as E, R and E' respectively. Taking the **plane of incidence** to be the plane formed by the incident ray and the normal to the surface at the point of incidence, we can also define **p-waves** as those waves with their electric vector vibrating parallel to the plane of incidence and **s-waves** as those waves with their electric vector vibrating perpendicular to the plane of incidence. Fresnel's laws may be stated as follows:

$$\frac{R_s}{E_s} = -\frac{\sin(\theta-\theta')}{\sin(\theta+\theta')} \qquad \frac{R_p}{E_p} = \frac{\tan(\theta-\theta')}{\tan(\theta+\theta')} \tag{58.6a}$$

$$\frac{E'_s}{E_s} = \frac{2\sin\theta'\cos\theta}{\sin(\theta+\theta')} \qquad \frac{E'_p}{E_p} = \frac{2\sin\theta'\cos\theta}{\sin(\theta+\theta')\cos(\theta-\theta')} \tag{58.6b}$$

The reflectances are then given by

$$r_s = \frac{R_s^2}{E_s^2} \quad \text{and} \quad r_p = \frac{R_p^2}{E_p^2} \tag{58.7}$$

First, with these equations we can answer the question of how much light is reflected at normal incidence. For the s-wave, we simply expand the sines to get

$$\frac{R_s}{E_s} = -\frac{\sin(\theta-\theta')}{\sin(\theta+\theta')} = \frac{\sin\theta\cos\theta' - \cos\theta\sin\theta'}{\sin\theta\cos\theta' + \cos\theta\sin\theta'}. \tag{58.8}$$

Making use of Snell's law we then find for the **reflectance at normal incidence**

$$r_s = \left(\frac{n-1}{n+1}\right)^2. \tag{58.9}$$

At normal incidence, it may be noted that the parallel and perpendicular components of the reflected light must be equal since the plane of incidence is undefined and the two components are indistinguishable. For r_p the process is a little tricky. Just setting $\theta = 0$ gives an indeterminate result but we can let both θ and θ' be small so that the tangents and sines are nearly equal. Taking this approach and setting the sines equal the tangents in equation 58.6a(a) we obtain the same result. As an example of the application of this result, suppose the index of refraction is 1.50. In this case, the reflectance $r = 0.04$ so that only 4% of the light is reflected.

58.6 External Reflection

Next, we can apply Fresnel's equations to obtain the polarizing angle for light incident on a more dense medium by plotting the reflectance for both s- and p-waves versus the angle of incidence as illustrated in figure **??**. The p-wave reflectance reaches a minimum value of 0 at about 56 degrees while the s-wave reflectance reaches about 15% reflectance at 56 degrees. This angle is called the **polarization angle** first discovered in 1808 by **Etienne-Louis Malus** (1775-1812), a French military officer, engineer, physicist, and mathematician. At the polarization angle, all the reflected light is polarized vibrating parallel to the plane of incidence. That the reflected and refracted rays are 90 degrees apart was first discovered in 1815 by Sir Daniel Brewster (1781-1868), a Scottish physicist. His discovery became known as **Brewster's law**, which may be obtained by noting that $\theta' + \theta = 90^o$ and setting $\sin\theta' = \cos\theta$ in Snell's law.

$$\tan\theta_c = n \tag{58.10}$$

Brewster's law shows that the polarizing angle depends only upon the index of refraction. Since the index of refraction varies somewhat with wavelength there will be some dispersion over the range of visible light. It is important to understand why light is polarized at this angle.

The incident light ray sets the electrons in the atoms of the material into oscillation and radiation from these atoms generates the reflected beam. Since the restoring forces are less in a direction perpendicular to the plane of incidence, the radiation from the atomic electrons is polarized perpendicular to the plane of incidence and the s-waves are reflected. This is the same reason that radiation from a horizontal antenna is polarized to vibrate in the direction of the antenna. The p-waves vibrating parallel to the plane of incidence are not absorbed by the atomic electrons near the surface and are transmitted into the reflecting medium.

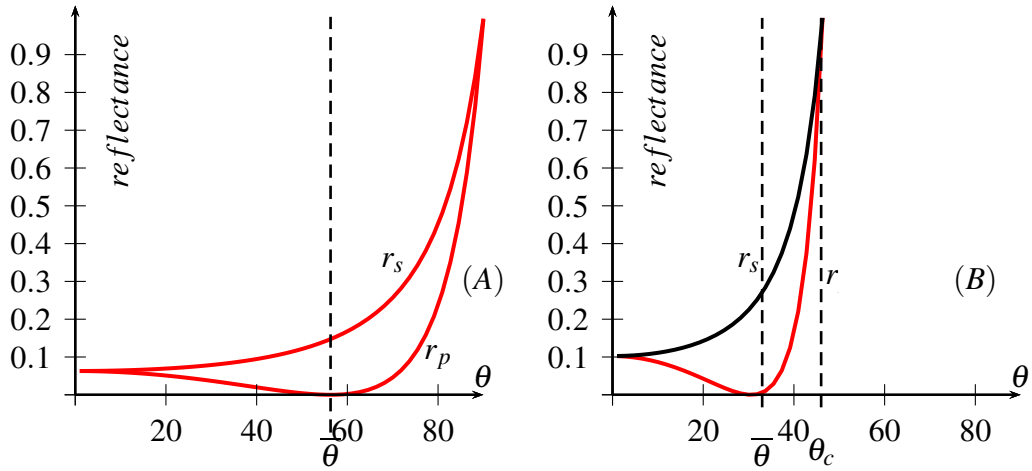


Figure 9.17: External (A) and Internal (B) reflectance versus angle of incidence.

The refracted waves can therefore be polarized parallel to the plane of incidence. As we see illustrated in figure 58.6(A), 15% of the s-wave radiation is reflected; therefore 85% must be transmitted. At the polarizing angle 0% of the p-waves are reflected and 100% are refracted.

The curves in figure 58.6(A) also illustrate that at normal incidence about 4% of the light is reflected for both p- and s-waves while at an angle of incidence of 90 degrees both p- and s-waves are totally reflected.

58.7 Internal Reflection

In a similar manner we can investigate **internal reflection** when the ray of light is incident from inside the dense medium and is refracted into the less dense medium by inverting the index of refraction as illustrated in figure 58.6(B). In this case, we see that the curves for p- and s-wave reflectance resemble those for external reflection beginning at about 4% reflection for both waves at normal incidence, $\theta = 0^o$, reach a polarizing angle at about 33 degrees for p-wave reflection but have a critical angle of about 41 degrees where internal reflection becomes 100%. The internal polarizing angle of 33 degrees corresponds to a refraction angle of 57 degrees. When the angle of incidence exceeds the critical angle of 41 degrees, there is total reflection inside the more dense medium. This critical angle can also

be calculated from Snell's law by setting θ'=0.

$$\sin\theta = \frac{1}{n} \tag{58.11}$$

In the case of $n = 1.50$, the calculated value of θ_c is 41 degrees. Brewster's law also gives a corresponding value for the internal polarization angle.

$$\tan\theta = \frac{1}{n} \tag{58.12}$$

which for $m = 1.50$ is θ = 33 degrees.

In one more application of Fresnel's laws, we can take up the questions of **phase changes on reflection**. Referring to equation 58.6a(a) we see that the sign of s-wave reflectance r_s is negative. Therefore a change of 180 degrees in phase will occur in the case of external reflection. For p-wave reflection, the sign of r_p is positive so that no change in phase will occur until the polarization angle is reached and $\theta + \theta'$ = 90 degees. At this point the sign of the denominator in equation 58.6a(b) changes so that the phase of p-waves also shifts by 180 degrees after the polarizing angle is exceeded. Up to the critical angle θ_c, the phase changes for internal reflection are exactly the reverse of those for external reflection. This result was anticipated by **Sir George Stokes** (1819-1903), an English scientist at Pembroke College, using the principle of reversibility, according to which the result of an instantaneous reversal of all the velocities in a dynamic system is to cause the system to retrace its previous motion. It is also a matter of experimental observation that in the reflection of light a phase change of 180 degrees occurs when light strikes the boundary from the side of higher velocity.[15] The conclusive and simple derivation of these phase changes as a consequence of Fresnel's laws demonstrates the advantage of a mathematical approach to analyzing physical phenomena.

58.8 Dispersion

Before leaving this section, a precise definition of **dispersion** is needed. The first evidence of dispersion usually encountered is observation of a beam of "white" light braking up into beams of colored light when it is refracted through a piece of glass. This is called dispersion of a beam of visible light into its component

[15]Francis A. Jenkins and Harvey E. White "Fundamentals of Optics" McGraw-Hill Book Company, New York (1957) p. 209

wavelengths. Returning to Snell's law, equation 58.2, it is apparent that the index of refraction must be a function of wavelength for dispersion to take place.

In figure 9.18 white light is dispersed into colors ranging from red to blue at the extremes of the dispersion. Each color will have its own refractive index. In crown glass for example, $n_R = 1.52441$, $n_B = 1.53303$ and $n_Y = 1.52704$. The **dispersive power** of the glass is defined as

$$\frac{1}{\nu} = \frac{n_F - n_C}{n_D - 1}, \tag{58.13}$$

where ν lies between 30 and 60 for most optical glasses. For crown glass the dispersive power is 0.0163 and $\nu = 61.28$.

Because of the apparent dependence of the index of refraction on the wavelength of light, several attempts were made to obtain an equation relating the two. One equation obtained by Sellmeier in 1871 is usually written[16]

$$n = \sqrt{1 + \sum_i \frac{A_i \lambda^2}{\lambda^2 - \lambda_i^2}}, \tag{58.14}$$

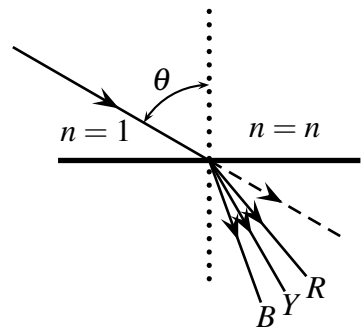


Figure 9.18: Dispersion of white light at a boundary.

where the λ_i correspond to the natural frequencies of vibration of the optical medium and the constants A_i are proportional to the number of atomic oscillators capable of vibrating at the natural frequency λ_i. Sellmeier's equation works very well except in the region of resonance at $\lambda = \lambda_i$ where sellmeier's equation yields an infinite result. This problem was resolved by the German physicist, H. L. F. Von Helmholtz (1821-1894), in a much more complex equation which takes account of the absorption of energy by assuming a frictional force proportional to the velocity of the atomic oscillator. Helmholtz's equation, which will not be quoted here, reduces to Sellmeier's equation when the wavelength is not equal the natural frequencies.

Problems

234. Starting with the relation $\omega = \frac{ck}{n}$, show that the group and phase velocities are related by $V_g = V_P - \lambda \frac{dV_P}{d\lambda}$.

[16] W. Sellmeier, Ann. Phys. Chem. 143, 271 (1871)

235. For the case of white light of wavelength 5890 Angstroms incident upon the surface of water, which has an index of refraction of 1.33, at an angle of incidence equal to 50 degrees, calculate the angle of reflection, angle of refraction, wavelength of the light in water and wave velocity in water.

236. For water, with $n = 1.33$ calculate the percent reflectance and refractance at normal incidence, the polarizing angle and critical angle for internal reflection and state the phase change in each case.

237. Calculate the dispersive power for crown glass with $n_F = 1.53303$, $n_D = 1.52704$ and $n_C = 1.52441$ for blue, yellow and red light respectively.

238. For borosilicate crown glass, which has an index of refraction of 1.51124 for light at 5890 Angstroms, assume an angle of incidence equal to 40 degrees, calculate the angle of refraction, wavelength of the light in the glass, wave velocity in the glass, the percent reflectance and percent refractance at normal incidence, the polarizing angle and critical angle for internal reflection, and state the phase changes that take place.

239. For vitreous quartz, which has an index of refraction of 1.45845 for light at 5890 Angstroms, calculate the percent reflectance and refractance at normal incidence, the polarizing angle, the dispersive power and critical angle for internal reflection and state the phase change in each case.

59 Reflection and Refraction of Sound Waves

The reflection of sound waves at a hard surface generally follows the same rules as reflection of light, but with the exception that

1. The law of reflection for sound waves is the same as those for light waves. However, the average wavelength of sound is about 100,000 times that of light; therefore a mirror or lens used to produce a beam of sound waves must be much larger than the mirrors and lenses used in optical systems. The reflecting surface must also be smooth compared to the wavelength of all sound waves being reflected.

2. When a sound wave strikes a hard surface there is no phase change.

3. When a sound wave from a more dense medium strikes a softer surface there is a phase change of 180 degrees.

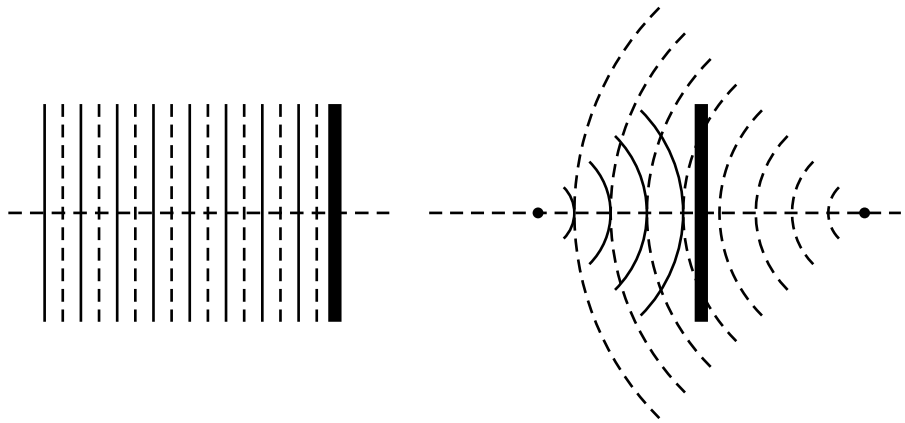


Figure 9.19: Sound waves with parallel wavefronts and sound waves from point source reflecting from smooth wall.

4. When a sound wave from a point source is reflected from a smooth surface, the waves radiate back as if they were coming from a virtual source equally distant but behind the reflecting surface.

Reflection of sound waves off of curved surfaces leads to different results. A concave surface tends to concentrate the reflected sound waves at some point. If the approaching waves are parallel wavefronts moving along the axis, the reflected waves will be focused at a point half-way between the radius of curvature and the reflecting surface. If the waves come from a point source they will be focused at some point between the radius of curvature and focal point of the mirror. The sound will be amplified at the focal point. Since the waves retrace their path when a signal is generated at the focal point a speaker can use a concave reflector to project his voice to an audience. The same principle is used in satellite disks to gather electromagnetic waves and focus them at a point where the signal receiver is located.

Sound reflected from a convex surface is diffused outward as if it came from a virtual source behind the reflecting surface. Cylindrical reflectors can therefore be placed at the corners of a room to diffuse sound throughout the room and give a more even sound.

Sound waves are diffracted as they pass through openings but the effect depends on the width of the opening and the wavelength of the sound wave. If the wavelength of the sound wave is large compared to the width of the opening, the refracted sound will spread out on the opposite side of the opening as if it were originating at a point source located at the opening. If, on the other hand, the

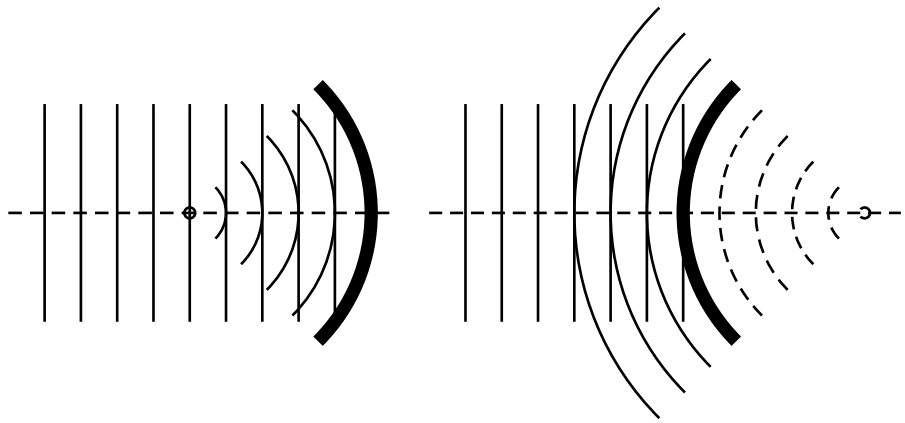


Figure 9.20: Sound waves reflecting from concave and convex surfaces.

wavelength of the sound waves is small compared to the opening, the wavefronts will pass through the opening with only their length truncated by the width of the opening. Dispersion of sound waves can also occur in passing through an opening if some of the sound waves are short compared to the opening and some are long. Because of refraction, sounds may be modified in passing through openings in cases where some of the sound waves have wavelengths long comparable to the opening with the long wavelengths being refracted more.

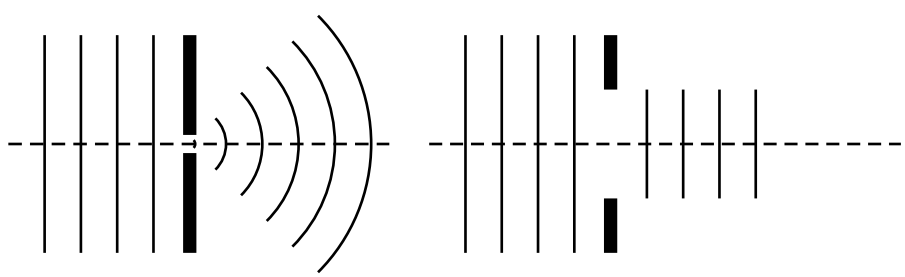


Figure 9.21: Sound waves refracted at apertures small and large compared to wavelength.

Referring to equation 56.8 we see that the velocity of sound increases with temperature. On a normal day, however, the temperature normally decreases with altitude as indicated in equation 22.12. This is called **temperature lapse** and amounts to about 6.5 $^oC/km$. Therefore the velocity of sound normally decreases

with altitude on a normal day as described by the following equations.

$$V = 331 + 0.61T(^oC) \tag{59.1a}$$
$$T(^oC) = 15 - 0.0065y \tag{59.1b}$$
$$V = 340 - 0.0039y \tag{59.1c}$$

For this reason, wavefronts of sound waves normally bend upward as they travel outward from a source near the ground as illustrated in figure 9.22. This effect may be viewed as a form of refraction of sound waves. As a result it may be difficult for someone standing at 100 meters away to hear when you shout at him much the same as it is possible for an observer standing on a river bank to see fish near the bank although there is no direct line of sight between the observer and the fish.

However, nature conspires to change things in the early morning with a **temperature inversion**. In a temperature inversion, the air at ground level is cool and the temperature increases with height above the ground. As a result, the speed of sound increases with height and the wave fronts bend downward. Temperature inversions often happen at night after the sun goes down and the ground or water in a lake cools off quickly while the air above the ground remains warm. This downward refraction of sound is why you can hear the conversations of campers across a lake.

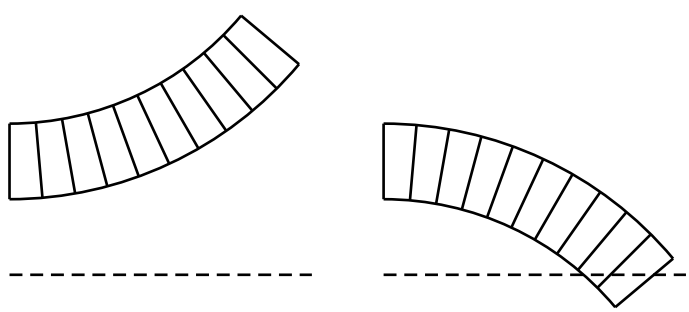


Figure 9.22: Sound waves with parallel wavefronts propagating across earth's surface during temperature lapse and temperature inversion (not to scale).

The last subject that relates to observable phenomena is that of reverberation and echo. Sounds will normally be reflected back towards the source in most room designs. The human brain tends to hold a sound for about 0.1 second so sounds reflected back within this time are interpreted by the brain as an extension or elongation of the original sound. This is called **reverberation**. Sounds that are

reflected back after about 0.1 second are normally perceived as a separate distinct sound. These are called **echos**.

In the late 1890s, Wallace Clement Sabine (1868-1919), an American physicist who founded the field of architectural acoustics, established an empirical relation between the reverberation time for the sound to fall 60 db, RT_{60}, the volume V and the surface area S of a room for an absorption coefficient of a between 0 and 1 for the room depending on the construction material.

$$RT_{60} \simeq 0.16\frac{V}{S} \quad sec \tag{59.2}$$

Although this equation does not take account of the shape of the room and cannot be considered an exacting relationship, it does provide a useful first approximation to the reverberation time of a room. An example of the reverberation times for several rooms calculated with this formula is provided in table 9.5.

Room Type	a	L	H	W	V	S	RT_{60}
Auditoriums	1	20	10	20	4000	1600	0.40
Concert Halls	1	30	20	30	18000	4200	0.69
Standard Rooms	1	5	2.5	5	62.5	100	0.10
Bathrooms	1	2	2.5	2	10	28	0.06
Bedrooms	1	5	2.5	8	100	145	0.11

Table 9.5: Reverberation times for typical constructions

This table, although only approximate, makes clear the difference between auditoriums and rooms inside dwelling houses. Auditoriums and concert halls will generally reflect sound as echos but the distances involved may reduce the level of the echo. Rooms in dwelling houses fit into the category of reverberation. While reverberation times near 0.1 second may be pleasing for certain types of music but may hinder clarity when precise understanding is important.

240. The design of an auditorium 20 meters wide includes a concave, reflecting wall behind the speaker. What must the radius of the concave surface be and how far from the wall must the speakers podium be placed. ans. 10m, 5m

241. Derive the equation relating the velocity of sound in air to altitude using $\gamma = 1.40$, $R = 8.31 Joules/mole/^oK$ and $T_o = 273.15\,^oK$.

242. Calculate the velocity of sound at 5000 meters on a normal day.

243. Calculate the velocity of sound 10 meters above a lake if temperature inversion increases the temperature from 20 to 30^oC.

244. What is the reverberation time for a room having a volume of 100 cubic meters and surface area of 80 square meters? ans. 0.20 seconds

60 Diffraction of Electromagnetic Waves

Diffraction through narrow apertures occurs only when the wavelength of the wave is comparable to the width of the slit. Diffraction phenomena are divided into two categories, those in which the source of light and screen on which the diffracted image is displayed are effectively at infinite distances from the aperture and those in which either the source or screen is at a finite distance from the aperture. Diffraction under the first condition is called **Fraunhofer diffraction** while diffraction under the second condition is called **Fresnel diffraction**. We will discuss four types of diffraction, (1) single slit diffraction, (2) double slit diffraction, (3) diffraction gratings and (4) diffraction from a crystal lattice. To investigate diffraction, we will represent the displacement of the electromagnetic vectors by an equation of the type developed in the previous sections.

60.1 Single Slit Diffraction

A typical diffraction experiment from a single aperture is illustrated in figure 9.23 in which the width of the aperture is b. The intensity of the diffracted rays is plotted on the screen a distance x behind the slit. A central maximum occurs along the x-axis and successive minima to each side of the slit. We will view the diffraction pattern as the sum of secondary waves generated at every point of the wavefront as it encounters the slit. The intensity of the image on the screen can be represented as the sum of two waves each from a element of width ds of the slit an equal distance from the axis. The amplitude of a wave emminating from the center of the slit can be represented by

$$y = A(x)\sin[\omega t - kx] \tag{60.1}$$

where x is the distance from the center point of the slit to the point P in figure 9.23 and A(x) is the amplitude dependent on the distance the wave travels from the

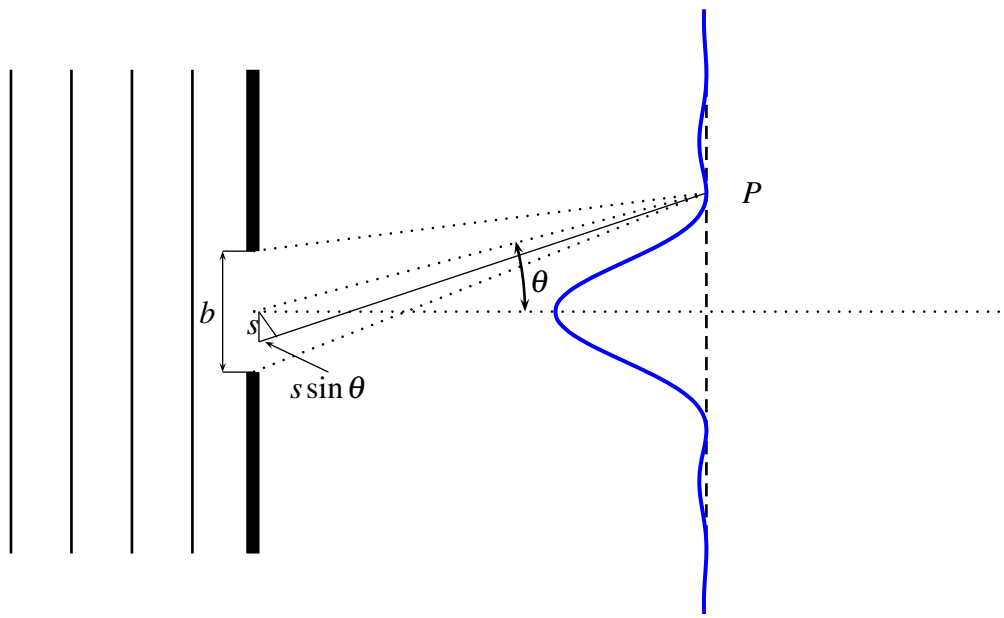


Figure 9.23: Diffraction of light from a single slit.

slit. The function A(x) is proportional to $1/\sqrt{x}$ for a cylindrical wave that emminates from a narrow slit and to $1/x$ for a spherical wave that emminates from a circular hole. Since we are seeking only the points of maxima and minima in the diffraction pattern the exact form of A(x) will not enter into the discussion. We can approximate the waves from an element ds at a distance s above and below the axis by

$$dy_{+s} = Ads\sin[\omega t - kx - k\delta] \tag{60.2}$$
$$dy_{-s} = Ads\sin[\omega t - kx + k\delta] \tag{60.3}$$
$$\tag{60.4}$$

where $\delta = s\sin\theta$. We find that the sum is[17]

$$dy_s = Ads[2\cos(ks\sin\theta)\sin(\omega t - kx). \tag{60.5}$$

Integrating over the range of s gives the total amplitude as a function of the

[17] After making use of the trigonometric identity $\sin\theta + \sin\phi = 2\cos\frac{1}{2}(\theta-\phi)\sin\frac{1}{2}(\theta+\phi)$

angle θ.

$$y = 2A\sin(\omega t - kx)\int_0^{b/2}\cos(ks\sin\theta)ds \tag{60.6}$$

$$= A_o\frac{\sin\beta}{\beta}\sin(\omega t - kx) \tag{60.7}$$

where $\beta = \frac{1}{2}kb\sin\theta$ and $A_o = 2Ab$. The resulting diffraction pattern has an amplitude that varies with the angle θ and varies as

$$I = A_o^2\frac{\sin^2\beta}{\beta^2} \tag{60.8}$$

Minima occur at values of $\beta = n\pi$ or at angles

$$\theta_m = \frac{m\lambda}{b}, \tag{60.9}$$

where m is a integer, and maxima occur in the diffraction pattern occur at values of

$$\tan\beta = \beta. \tag{60.10}$$

One important observation from equation 60.9 is that the angular width of the central peak is inversely proportional to the width of the slit b so that the width of the central peak increases as the slit is made smaller and the width decreases as the slit is made larger. This phenomena will play an important role in atomic physics. Lord Rayleigh considered the angle of first minima and θ_1 in equation 60.9 to be the **resolving power** of the aperture since two sources of light with an angular separation θ_1 would have central peaks that occurred at the minima of the diffraction pattern for each source. This definition is still in use.

The results are essentially the same for circular apertures, except that the resolving power becomes

$$\theta_1 = 1.22\frac{\lambda}{D}, \tag{60.11}$$

where D is the diameter of the aperture. This resolving power is used as the **minimum angle of resolution** for telescopes. Taking the effective wavelength of white light as 5×10^{-5} and the diameter of the pupil of the human eye as 0.3 cm, we find that the human eye has an angular resolution of 2×10^{-4} radians or 42 seconds of arc. A very large refracting telescope with a diameter of 40 inches would have an angular resolution of 0.14 seconds.

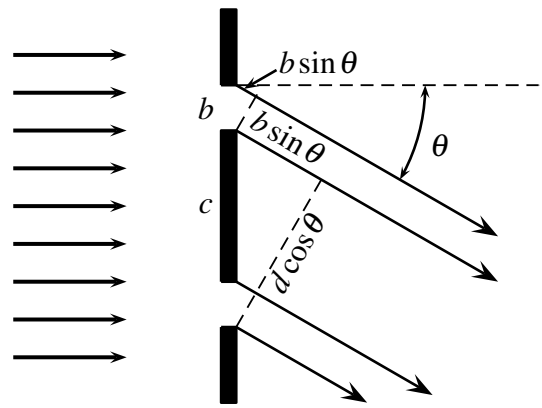


Figure 9.24: Diffraction of light through double slits.

60.2 Double Slit Diffraction

The interference of light passing through two narrow, side by side apertures was demonstrated in 1801 by Sir Thomas Young, an English physician and physicist, in an attempt to determine whether light was composed of particles or waves.

As illustrated in figure 9.24 the two slits are of width b separated by a distance c, thus allowing us to define

$$d = b + c \tag{60.12}$$

$$s_1 = \frac{1}{2}(d - b) = \frac{1}{2}c \tag{60.13}$$

$$s_2 = \frac{1}{2}(d - b) = b + \frac{1}{2}c \tag{60.14}$$

The interference patter is the combination of the single slit pattern and the double slit pattern. We can calculate the intensity of the diffraction pattern by following the same procedure as for the single slit but changing the limits of integration from s_1 to s_2. This results in an equation for the amplitude similar to that of equation 60.7.

$$y = 2bA\frac{\sin\beta}{\beta}\cos\gamma\sin(\omega t - kx) \tag{60.15}$$

where the parameters β and γ are defined as

$$\gamma = \frac{1}{2}k(b+c)\sin\theta = \frac{\pi}{\lambda}d\sin\theta \tag{60.16a}$$

$$\beta = \frac{1}{2}kb\sin\theta = \frac{\pi}{\lambda}b\sin\theta \tag{60.16b}$$

The intensity of the double slit diffraction pattern can then be written as

$$I = 4A_o^2\frac{\sin^2\beta}{\beta^2}\cos^2\gamma \tag{60.17}$$

where $A_o = 2bA$. The complete double slit diffraction pattern is the product of two factors. The broad envelope represented by the dotted line results from interference of the beams from the two slits and is plotted from the $\cos^2\gamma$ factor. The narrow maxima represented by the shaded peaks results from diffraction of light by either slit and is plotted from the $\frac{\sin^2\beta}{\beta^2}$ factor. The double slit pattern is therefore a combination of interference and diffraction.

A graphical examination of the effect that results from changing the separation distance between the apertures as illustrated in figure 9.25 will clarify the picture. In the first three cases, $b = 5\lambda$ while the separation between the apertures is taken as $c = 10\lambda$, $c = 15\lambda$ and $c = 20\lambda$. Examination shows that the number of interference peaks increases as the distance between the slits is increased while the diffraction envelope remains stationary. The fourth graph results from increasing the width of the slits with the spacing between them held at the same value as in the third graph. It may therefore be concluded that the width of the envelope decreases and encompasses fewer of the interference fringes as the width of the slits is increased. Based on these effects, double slit diffraction instruments may be designed to investigate phenomena of different types.

Now, we are in a position to investigate the separation of maxima and minima from a mathematical standpoint. To begin, the double slit diffraction pattern has minima whenever the conditions for a minimum occur in the interference pattern. From equation 60.17, this is seen to occur whenever $\gamma = \pi/2$. From the definition of γ in equation 60.16(a), **interference minima** occur when

$$d\sin\theta = \left(n+\frac{1}{2}\right)\lambda \quad \text{where } n = 0,\ 1,\ 2,\ 3,\cdots. \tag{60.18}$$

The double slit diffraction pattern also has a minima whenever the conditions for a minimum occur in the diffraction pattern, which occurs at $\beta = n\pi$ for n a whole integer but not zero as seen from equation 60.17. Therefore from equation 60.16(b),

diffraction minima occur when

$$b\sin\theta = m\lambda \quad \text{where } m = 1,\ 2,\ 3, \cdots. \tag{60.19}$$

Maxima occur whenever $\gamma = n\pi$ as seen from equation 60.17 irregardless of the value of β. Therefore the occurence of maxima are determined by interference of waves diffracted from the two slits and the defining condition for a maxima becomes

$$d\sin\theta = n\lambda \quad \text{where } n = 0\ 1,\ 2,\ 3, \cdots \tag{60.20}$$

In these equations, n represents the number of wavelengths in the path difference between corresponding points in the two slits and therefore represents the *order of interference*. When the conditions for a maxima and a minima are both satisfied for the same value of θ, the orders of interference will be missing. The missing orders occur whenever

$$d\sin\theta = n\lambda \tag{60.21}$$

$$b\sin\theta = m\lambda \tag{60.22}$$

$$\frac{d}{b} = \frac{n}{m} \tag{60.23}$$

60.3 Diffraction Gratings

A **diffraction grating** consists of a large number of slits lying parallel to one another and equidistant from one another. The effect of diffraction gratings on parallel light was first investigated by Fraunhofer in 1819, who prepared his gratings by winding fine wires around two parallel screws. The most important effect of a diffraction grating is to increase the sharpness of the interference peaks plotted from the $\cos^2\gamma$ factor in the double slit experiment. As the number of gratings increases, the diffuse pattern of the double slit sharpens rapidly and after about 20 gratings is reached the diffraction pattern becomes a series of sharp lines.

The intensity of the diffraction pattern from a grating results from individual contributions from each slit which are of equal magnitude but arrive at at different times represented by a phase factor δ. For a diffraction grating with N slits of

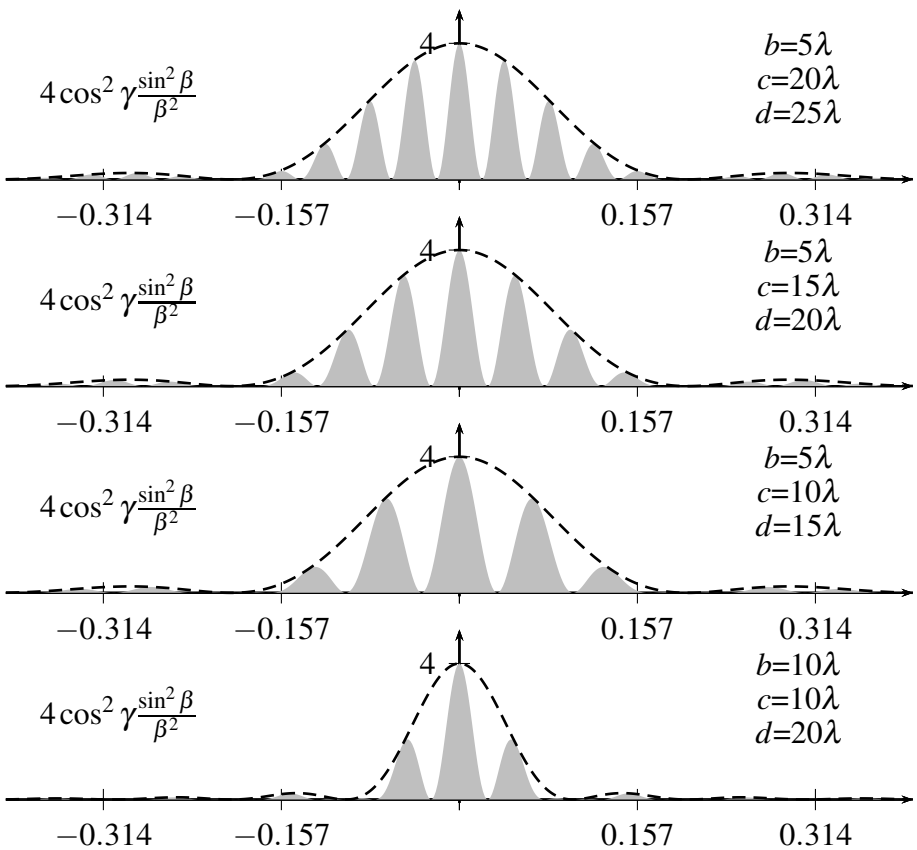


Figure 9.25: Diffraction pattern for a double slit plotted as intensity versus diffraction angle.

width b and separation c, the formula for the intensity is

$$I = A_o^2 \frac{\sin^2\beta}{\beta^2}\frac{\sin^2 N\gamma}{\sin^2\gamma} \quad \text{where} \tag{60.24a}$$

$$\beta = \frac{1}{2}kb\sin\theta = \frac{\pi}{\lambda}b\sin\theta \tag{60.24b}$$

$$\gamma = \frac{1}{2}k(b+c)\sin\theta = \frac{\pi}{\lambda}d\sin\theta \tag{60.24c}$$

As before $d = b + c$, A_o is the amplitude contributed by a single slit, $\frac{\sin^2\beta}{\beta^2}$ is the diffraction term for light from each slit and $\frac{\sin^2 N\gamma}{\sin^2\gamma}$ is the interference term for

the N slits. If we set $N = 2$, equation 60.24(a) reduces to the intensity for a double slit experiment.

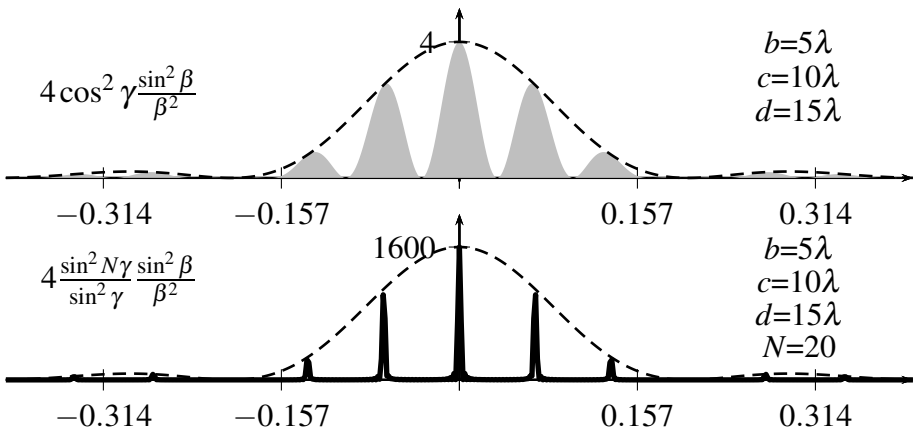


Figure 9.26: Comparison between the diffraction patterns of a double slit and a grating with 20 slits.

A comparison between the double slit diffraction pattern in the third graph of figure 9.25 and a diffraction pattern for a grating with the same slit width and slit spacing but with 20 slits is illustrated in figure 9.26. As can be seen by comparing these two graphs, the grating with 20 slits narrows the interference peaks to resemble lines but retains the same spacing between the peaks and the same relative intensity. Therefore, it is easy to see that the grating greatly improves the resolving power of a double slit.

Other features may also be noticed. One is that the amplitudes of the spectral lines are N^2 times larger than for the double slit. This is because in the limit of large γ, the interference term $\frac{\sin^2 N\gamma}{\sin^2}$ approaches $\pm N$.

Another is that the angles at which the spectral peaks appear can be described by

$$d\sin\theta = m\lambda \quad \text{where } n = 0\ 1,\ 2,\ 3, \cdots \tag{60.25}$$

Equation 60.25 is often referred to as the **grating equation**. The integer m is also called the **order number**. The interference term requires that the intensity of the spectral peak be zero at $N\gamma = p\pi$ unless $p = mN$ at which the grating equation requires a maximum. Thus there are N-1 minima between spectral peaks.

By differentiating the grating equation, we can obtain a formula for the **angular dispersion** of two wavelengths.

$$\frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta} \tag{60.26}$$

Because the wavelength appears in the formula $\gamma = \frac{\pi d}{\lambda}\sin\theta$ the diffraction grating will also separate waves of different wavelengths. The normal criteria for separation is the wave of one wavelength must have its maximum at the minimum of the other wavelength. Stated mathematically, this criteria reads

$$mN\lambda + \lambda = mN(\lambda + \Delta\lambda), \tag{60.27}$$

which allows to define the **resolving power** of a grating as

$$R = \frac{\lambda}{d\lambda} = mN. \tag{60.28}$$

Thus we see the resolving power is zero for the central maxima as we would expect it to be and increases with order number giving the greatest separation of wavelengths at the largest order.

60.4 Diffraction from a Crystal Lattice

Crystalline solids are those in which the atoms are arranged in some definite order that is constantly repeated. There are many crystalline forms the most common of which are the cubic, tetragonal, hexagonal, orthorhombic, monoclinic and triclinic. The property important for diffraction of electromagnetic waves is that the crystal lattice spacing is of the order of angstroms, which is convenient for diffraction of X-rays.

The diffraction of X-rays from a simple crystal lattice is illustrated in figure 9.27 in which the lattice spacing is d and the angle of incidence relative to the crystal plane is θ. It can be seen from this illustration that each ray reflected from successively lower crystals planes must travel a greater distance $2d\sin\theta$ that the ray above it. Thus the condition for reinforcement is

$$n\lambda = 2d\sin\theta. \tag{60.29}$$

This equation, called **Bragg's law** and named after the English physicists Sir W.H. Bragg and his son Sir W.L. Bragg who derived it in 1913, gives the relationship

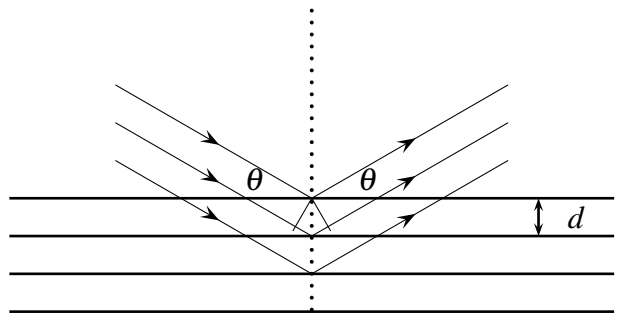


Figure 9.27: Diffraction of X-rays by a crystal lattice.

between the lattice spacing and the angle of reflection. The reflection corresponding to $n = 1$ is first order, $n = 2$ second order, and so on. The X-rays are actually refracted since they are only reflected at certain angles determined by the lattice spacing and wavelength of the X-ray.

Bragg's law may be used to measure the wavelength of X-rays. For example, Sodium chloride has a spacing of 2.82×10^{-10} meters and X-rays reinforce constructively at a diffraction angle of 5.9 degrees. Therefore from Bragg's equation

$$\lambda = 2(2.82 \times 10^{-10}) \sin 5.9^o = 5.81 \times 10^{-11} \quad \text{meters} \tag{60.30}$$

Problems

245. For single slit diffraction, find the angle at which the first minimum occurs, the angle at which the second maximum occurs and the percentage intensity of the second, third and fourth maxima relative to the central maxima when light from a Mercury arc of wavelength 5461 angstroms is incident on a slit of width 0.2mm. ans. 0.00273 rad, 0.0039 rad, 4.72%, 1.65% and 0.83%

246. Assume that the diameter of the pupil of the human eye is 2.5mm. At what distance would two orange colored objects emminating light of 6000 angstroms, 40 cm apart, be resolved by diffraction of light passing through the pupil of the eye if the criteria for resolution is that the minimum of the diffraction pattern of one object falls on the second. ans. 1.37 km

247. In double slit diffraction, experiment using the 5461 angstrom line from Mercury and a diffraction grating with b=0.200 mm and c=0.800 mm, what orders are missing from the spectrum; what is the ratio of the intensity of

the third order to the first and at what angle does the third order appear? ans. 5, 10,15,..., 0.254, 0.0938 degrees.

248. How many interference maxima appear under the central diffraction maximum for the case of d/b=4 and what are their relative intensities? ans. 7, 1, 0.810, 0.405, 0.090

249. If a diffraction grating with N=20 and d/b=3 is used, which orders will be missing, how many numbers will there be under the peak and what will their relative intensities be? ans. 3, 6, 9, 12, $\cdots$, 5, 1.00, 0.685, 0.172, 0.00

250. Suppose that the diffraction grating has N=50 and d/b=10, which orders will be missing, how many numbers will there be under the peak and what will their relative intensities be? ans. 10, 20, 30,$\cdots$, 49, 1.00, 0.968, 0.875, 0.737, 0.573, 0.405, 0.255, 0.135, 0.055, 0.012, 0.00

251. If X-rays reinforce constructively when diffracted from a crystal with lattice spacing of 3.000 angstroms at an angle of 3.125 degrees, what is the wavelength of the X-rays? ans. 0.327 angstroms

Chapter 10

MODERN PHYSICS

In the previous chapters we examined the fundamental principles of physics that underlie our studies in any branch of physics. Most of the concepts we studied were discovered and documented in the nineteenth century. No text would be complete without following these concepts into the twentieth century and documenting some of the major developments that arose and led to the development of modern quantum physics. This will be subject of this chapter.

61 Electromagnetic Radiation

The classical theory of electromagnetic radiation was presented in the last chapter and will be briefly summarized here pointing out the features that became important in the development of modern physics. In 1864, James Clerk Maxwell described the electromagnetic field as a continuous vector field with electric and magnetic components that obey four equations. These equations became known as Maxwell's equations. They are linear differential equations are first order and describe in a general way the relationships between the vector field strengths, charge density and current density.

$$\nabla \cdot \mathbf{D} = \rho \tag{61.1}$$

$$\nabla x \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{61.2}$$

$$\nabla x \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{61.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{61.4}$$

These equations led to the wave equation for electromagnetic field and were used to explain all electromagnetic phenomena that have been observed.

$$\nabla^2\mathbf{E} - \frac{\kappa\kappa_m}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \mu\sigma\frac{\partial\mathbf{E}}{\partial t} = 0 \tag{61.5}$$

$$c^2 = \frac{1}{\mu_o\varepsilon_o} \tag{61.6}$$

In a non-conducting medium, the third term is zero, leaving a wave equation for waves that travel with a phase velocity $1/\sqrt{\varepsilon\mu}$. In a conducting medium, the second term is usually negligible; leaving the same differential equation that describes heat conduction or diffusion. The constant c is written as the product of the two free space constants μ_o and ε_o so that it must have the dimensions of velocity L/T. This constant represents the velocity of the propagating waves and is the velocity of light in a vacuum. Electromagnetic theory differs from classical mechanics in that the velocity of light in a vacuum appears as a fundamental constant of the theory. By the late 1800s, Maxwell's wave equations came to dominate the theory of electromagnetism and were successful in describing the behavior of electromagnetic waves in numerous applications.

61.1 Failure of Classical Theory

As it gained acceptance in explaining electromagnetic phenomena at the macroscopic level, theoreticians began to use the classical theory of electromagnetism to explain experimental observations at the atomic level. British Physicist J. J. Thomson used the classical theory in 1903 to explain scattering of light waves from free electrons in passing through gases or other transparent media.[1] The result, known as the cross section for Thomson Scattering, is (a) constant, (b) independent of the frequency of the incident radiation and (c) symmetric with respect to forward and backward scattering. A detailed derivation will show that the only dependence of the cross section is on polarization of the incident wave. Expressed in terms of the classical electron radius, $r_o = \frac{e^2}{mc^2} = 2.82x10^{-13}cm$, and the angle of scattering, θ, the differential and total Thomson scattering cross sections are as

[1] J. J. Thomson, *Conduction of Electricity through Gases*, New York, Macmillan, 2nd ed., (1903) p. 321

follows:

$$\frac{d\sigma_o}{d\Omega} = \frac{r_o^2}{2}\left(1+\cos^2\theta\right) \tag{61.7}$$

$$\sigma_o = \frac{8\pi}{3}r_o^2 = 6.6x10^{-25}cm^2 \tag{61.8}$$

$$\tag{61.9}$$

Thomson's formula was found to be correct at low energies between the binding energies of electrons and the rest mass energy of an electron. However, in 1923 A. H. Compton found that scattered wavelengths of higher energy X-rays scattered at various angles were shifted from that of the incident beam and the amount of the wavelength shift increased with scattering angle. This observation contradicted the Thomson cross section and indicated that understanding of electromagnetic radiation was still incomplete.

A second major challenge arose in explaining thermal radiation emitted by objects because of their temperature. The problem of explaining the energy spectrum of light waves emitted from a cavity, the so-called black body problem, attracted the most attention. The energy spectrum of radiation from a cavity could be measured by passing the light from the body through a prism and measuring the intensity at each wavelength. Classical theory indicated that an infinite amount of energy could be stored in a cavity and that radiated energy should increase as the third power of the frequency of the radiation. This result was clearly in contradiction to reality. The German physicist, Wilhelm Wien, attempted to explain the energy spectrum of radiation from a cavity in 1896, but obtained only a law that was valid at high frequencies. He did, however, obtain the displacement law for radiation that could be fitted to empirical data and earned him the Nobel prize in 1911. The British physicists, J. W. S. Rayleigh and J. H. Jeans, also attempted to explain the results of experimental observations, but obtained only a law that was valid at low frequencies. Failure of these theories had created serious problems for classical physics by 1900 because these failures called into question the basic concepts of thermodynamics and electromagnetism that were involved in the reasoning that led to these formulas.

Another problem arose in studying the ejection of electrons from a metallic surface by directing a light beam onto it. In the classical concept of electromagnetism, it would be expected that there would be a threshold in energy of the electromagnetic field for ejection of electrons and that both the number of electrons ejected and their energy would be in proportion to the intensity of the beam.

In 1902, the Hungarian Physicist, Philipp Lenard, observed that although the intensity of the ejected electrons was proportional to the intensity of the incident beam, the energy of individual emitted electrons increased with the frequency of the incident beam but the energy was not dependent of the intensity of the incident beam. This was in contradiction to Maxwell's wave theory of light, which predicted that the electron energy would only be proportional to the intensity of the radiation. Thus another failure of classical electromagnetism near the beginning of the 20th century.

The problem of cavity radiation was solved in 1900 by the German Physicist, Max Plank, who quantized the radiation in the cavity to obtain a law that correctly described the energy spectrum of radiation emitted from a cavity. For his work, Planck was awarded the Nobel Prize in Physics in 1918. In 1905, another German Physicist, Albert Einstein, explained ejection of electrons from metal surfaces by describing light as composed of discrete quanta with energy equal to the frequency multiplied by a constant. He then visualized only one photon interacting with one electron. As a result, the energy of the ejected electron was dependent only on the frequency of the incident radiation but the number of electrons ejected would depend on the intensity of the incident beam, creating the observed effect. There would be a threshold energy determined by how strongly the electron was bound to the surface of the metal. This discovery led to the quantum revolution in physics and earned Einstein the Nobel Prize in Physics in 1921. However, the problem of explaining scattering of light waves at high energies remained until 1923 when it was successfully described by the American Physicist Arthur Compton, who showed that the radiation must be quantized and the scattering treated as the classical mechanics scattering of a quanta with energy $E = h\nu$, and momentum, $p = E/c$ from an electron of mass m and velocity 0. Compton was awarded the Nobel Prize in Physics in 1927 for this work.

These experiments and their interpretation will be described in the following sections to illustrate the need to view electromagnetic radiation as having a particle nature in order to explain the observed phenomena.

61.2 Black Body Radiation

Near the end of the nineteenth century, the frequency spectrum of electromagnetic radiation from a black body had been measured and found to increase with frequency to a peak value and then drop asymptotically toward zero. All attempts to explain the observed spectra of blackbody radiation using classical theories failed. Agreement between theory and experiment was achieved only when Max Plank

treated blackbody radiation as a collection of photons with quantized energy.[2] The theoretical result, known as Plank's equation, was in complete agreement with the observed spectra. In this formula, h is Plank's constant, k is Boltzman's constant, c is the velocity of light, T is the temperature in degrees Kelvin, ν is the frequency of the electromagnetic radiation and $u(\nu)$ is the energy per unit volume in the frequency interval $d\nu$ between ν and $\nu + d\nu$.

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1} \tag{61.10}$$

From this formula for the energy density inside a cavity, the formula for energy radiated from a hole of unit area per unit time may be calculated.

$$w(\nu) = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1} \tag{61.11}$$

Plank's formula reduces to the Classical Rayleigh-Jeans radiation formula at low frequencies

$$w(\nu) = \frac{2\pi h\nu^3}{c^2} \tag{61.12}$$

and to Wien's formula at high frequencies.

$$w(\nu) = \frac{2\pi h\nu^3}{c^2}e^{-}h\nu/kT \tag{61.13}$$

Plank's formula can be integrated to obtain Stefan's law and an expression for Stefan's constant.

$$\int_o^\infty w(\nu)d\nu = \frac{4\sigma}{c}T^4 \tag{61.14}$$

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^3} \tag{61.15}$$

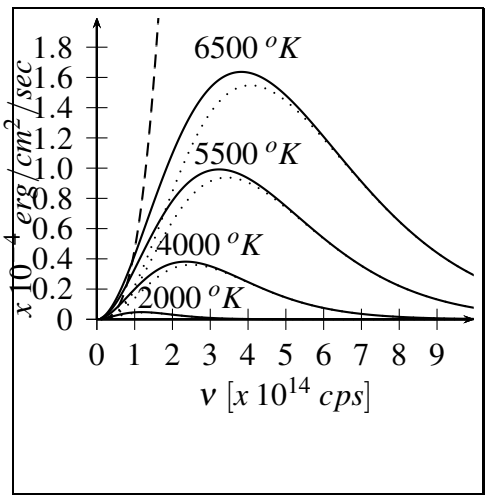


Figure 10.1: Frequency spectrum of radiation from a black body calculated from Plank's formula at different temperatures compared to the predictions of Wien ($\cdots$) and Rayleigh-Jeans ($- - -$).

[2] M. Plank, "Ueber das Gesetz der Energieverteilung im Normalspectrum," Ann. Physik 4, 553 (1901)

Plank's formula also yields a maximum energy density at a wavelength predicted by Wien's law.

$$\lambda T = b \tag{61.16}$$

The experimental values of the Stefan-Boltzman constant σ and Wien's constant b then allowed the value of Plank's constant h to be calculated.

$$h = 6.625x10^{-27} erg - sec \tag{61.17}$$

The significance of Plank's work is that it introduced the concept of quantizing energy levels to obtain agreement with experimental measurements. This opened the way for other theoreticians to speculate on solutions to experimental observations that could not be explained using classical concepts of physics and mathematics. Later, Albert Einstein derived a radiation law emphasizing statistical considerations instead of the details of the electromagnetic spectrum and introduced the concept of detailed balance for transitions induced by an electromagnetic field.

61.3 Photoelectric Effect

The photoelectric effect occurs when light is shined on a clean metallic surface. Although electrons are free to move about the metal, they are bound to the surface. Light shinning on the surface of the metal imparts energy to these electrons which may then be ejected from the surface. Heinrich Hertz first observed evidence of the photoelectric effect in 1887 while studying the properties of electromagnetic waves produced by oscillating electric currents. In 1902, Philipp Lenard found that: 1) photoelectrons were emitted from the surface of the metal immediately (within 10^{-9} seconds) upon light being shined on the metal, 2) the photocurrent increases linearly with the intensity of the light beam, 3) the kinetic energy of the photoelectrons increased linearly with frequency of the light waves after a certain

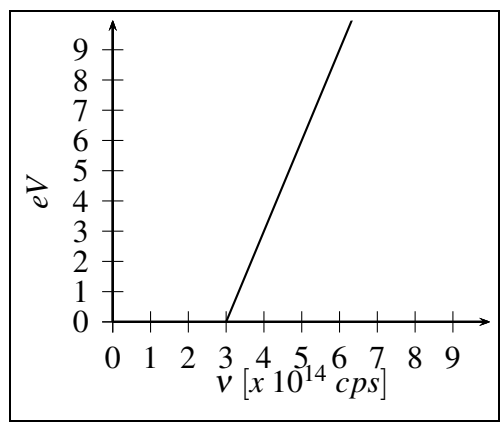


Figure 10.2: Typical energy spectrum of photoelectrons ejected from a metal having a work function of 1.25 eV

threshold frequency was reached and 4) the energy of the ejected electrons was independent of the intensity of the incident beam. This last discovery that attracted the most interest because it could not be explained on the basis of classical theory.

The slope of the line turned out to be the same for all metals and also the same as Plank's constant h. In 1905, A. Einstein applied quantum theory to explain the photoelectric effect by assuming that the light beam was composed of photons, each having an energy E that depended only upon the frequency ν and Plank's constant.[3]

$$E = h\nu \tag{61.18}$$

Then assuming that there was a minimum energy that the electron needed to escape the metal, Einstein constructed the photoelectric equation in which ν_o represented the frequency of photons having the minimum energy needed to cause ejection of electrons from the metal.

$$\frac{1}{2}mv^2 = h\nu - h\nu_o \tag{61.19}$$

This equation matched experimental data perfectly and demonstrated clearly that natural phenomena could be explained by assuming that light beams were composed of quanta of energy called photons. That the value of Plank's constant could also be obtained from the slope of the line was further confirmation of the theory.

61.4 The Compton Effect

From a classical standpoint, light rays scattered from an electron would be expected to have the same frequency as before striking the electron. This assumption proved false in experiments when the scattered light beam was found to have a smaller frequency than before being scattered. Classical theory could not explain this shift in wavelength. In 1923, A. H. Compton observed that the wavelengths of x-rays scattered in thin targets was less than the wavelength of the incident x-rays. He was able to show theoretically that this phenomena could be explained by using Einstein's approach to the photoelectric effect.[4] Compton assumed that the energy of each photon was taken as $E = h\nu$, and used Einstein's relativistic theory to assign a momentum $p = E/c$ to the photon.

[3]Einstein, Albert "On a Heuristic Viewpoint Concerning the Production and Transformation of Light" Annalen der Physik 17: 132-148 (1905).

[4]Compton, A. H.,"A Quantum Theory of the Scattering of X-Rays by Light Elements". The Physical Review 21 (5) (1923) pp. 483-502.

Then using the principles of conservation of energy and momentum in the collision between a photon and the electron, Compton derived an expression, called the Compton Effect Equation, for the shift in the wavelength of the scattered light as well as the energies of the scattered photon and recoiling electron.

$$\triangle\lambda = \frac{h}{m_o c}(1-\cos\theta) \qquad (61.20a)$$

$$E'_\gamma = \frac{E_\gamma}{1+\frac{E_\gamma}{mc^2}(1-\cos\theta)} \qquad (61.20b)$$

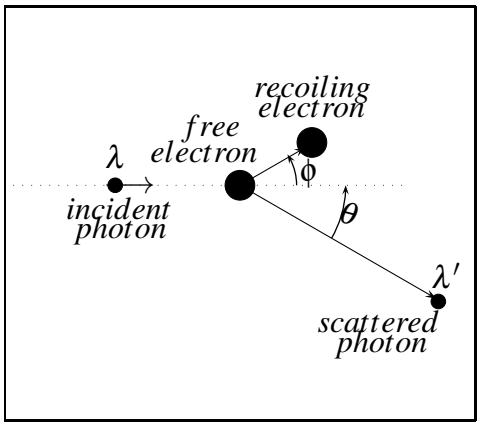


Figure 10.3: Gamma ray scattering through an angle θ from a free electron which recoils at an angle of ϕ. The wavelength of the gamma ray is shifted by an amount $\Delta\lambda = \lambda' - \lambda$.

Subsequent experiments verified Compton's theory. The constant $h/m_o c$ = 0.024263102175Å became known as the Compton wavelength and has the dimensions of length. In 1928, Klein and Nishina used quantum mechanics tocalculate the cross section for Compton scattering and found that it confirmed the shift in wavelength for the scattered x-rays and reduced to the Thomson scattering cross section at low energies.[5] Thus, once again, it became necessary to characterize electromagnetic waves as having a corpuscular nature.

Problems

252. The threshold for the photoelectric effect on a Sodium surface is 5400 A. If radiation of 2483 A from an intense spectral emission from Iron impinges on the Sodium surface, what will be the energy of the photoelectrons ejected. ans. 2.7 eV

253. What is the percentage shift in the wavelength of the 2483 line from iron in Compton scattering at 90.00 degrees? ans. 0.000977%

254. What is the shift in the wavelength of a incident radiation having wavelength 1.0 A in a Compton scattering at 180 degrees and the percentage shift? ans. 0.0485 A, 4.85%

[5] Klein, O; Nishina, Y (1929). Z. f. Phys. 52: 853 and 869

255. What is the maximum energy that can be imparted to an electron during a Compton scattering with an energetic photon? ans. $\frac{2h\nu/m_oc^2}{1+2h\nu/m_oc^2}$

62 Classical Mechanics

We previously studied classical mechanics from a non-theoretical standpoint. A short digression into a more formal embodiment of classical mechanics will prove useful in explaining further development of modern physics. For this purpose we must introduce the Lagrangian and Hamiltonian. Hamiltonian mechanics is a reformulation of classical mechanics that was introduced in 1833 by Irish Mathematician William Rowan Hamilton. It arose from Lagrangian mechanics, a previous reformulation of classical mechanics introduced by the French Mathematician, Joseph Louis Lagrange, in 1788. The interested reader is referred to a text by Abraham Becker for a more complete discussion of classical mechanics. For our present purpose, we may be content with a few simple definitions.[6]

For the first definition, the Lagrangian $L(q,\dot{q},t)$ defined as

$$L = T - V, \tag{62.1}$$

where T is the kinetic energy and V is the potential energy of the particle, allows Newton's equation of motion $F = ma$ to be replace by Lagrange's equation of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{62.2}$$

The Hamiltonian $H(p,q,t)$, defined by

$$H = T + V, \tag{62.3}$$

allows Newton's second law to be replaced by Hamilton's canonical equations,

$$H = \sum_m \dot{q_m} p_m - L \tag{62.4a}$$

$$\dot{q_m} = \frac{\partial H}{\partial p_m} \tag{62.4b}$$

$$\dot{p_m} = \frac{\partial H}{\partial q_m} \tag{62.4c}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{62.4d}$$

[6] R.A. Becker, Introduction to Theoretical Mechanics, Chapter 13, McGraw-Hill, New York, 1954

In these equations, the quantity q_m is considered the generalized coordinate and the quantity p_m is considered the generalized momentum. The generalized coordinates are those coordinates left in the equations of motion after the physical constraints and boundary conditions of the problem have been used to eliminate all other coordinates. To make these definitions seem more familiar, we can take a simple example of the clock pendulum as it was discussed in section 15.1. For the clock pendulum, the coordinates are (ℓ, θ) where ℓ is the length of the pendulum arm and θ is the angle between the vertical and the pendulum arm. Since the pendulum arm is fixed the remaining, and therefore the generalized, coordinate is θ. Then the kinetic and potential energies are given by

$$K = \frac{1}{2} m\ell^2 \dot{\theta}^2 \tag{62.5}$$

$$V = mg\ell(1 - \cos\theta). \tag{62.6}$$

This allows us to write the Lagrangian and Hamiltonian as

$$L = \frac{1}{2} m\ell^2 \dot{\theta}^2 - mg\ell(1 - \cos\theta) \tag{62.7}$$

$$H = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell(1 - \cos\theta) \tag{62.8}$$

From the Lagrangian, we can calculate

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ell^2 \dot{\theta} \tag{62.9}$$

$$\frac{\partial L}{\partial q} = mg\ell \sin\theta, \tag{62.10}$$

and use Lagrange's equation to find

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0, \tag{62.11}$$

exactly the equation of motion obtained in equation 15.2. We can also use Hamilton's canonical equations to find

$$\frac{\partial H}{\partial \theta} = mg\ell \sin\theta = m\ell\ddot{\theta} = \dot{p_\theta}. \tag{62.12}$$

This result is equal to the force accelerating the pendulum back toward is equilibrium position and therefore the mass of the pendulum times its acceleration or the

time derivative of its momentum, which is easily verified from the momentum of the pendulum $p = m\ell\dot{\theta}$. Without going further, the potential usefulness of these formulations in formulating theoretical mechanics should be apparent.

One benefit obtained immediately from equation 62.4 is that the total derivative of the Hamiltonian will be a constant of the motion when the Hamiltonian is not an explicit function of time. This observation will arise again in the study of quantum mechanics.

$$dH = \sum_m \left(\frac{\partial H}{\partial q_m} dq_m + \frac{\partial H}{\partial p_m} dp_m \right) + \left(\frac{\partial H}{\partial t} \right) \tag{62.13}$$

By defining the Poisson bracket of any two quantities that are arbitrary functions of the generalized coordinates,

$$(F,G) = \sum_m \left(\frac{\partial F}{\partial q_m} \frac{\partial G}{\partial p_m} + \frac{\partial F}{\partial p_m} \frac{\partial G}{\partial q_m} \right) \tag{62.14}$$

it is easy to see that Hamilton's equations of motion can be expressed in terms of Poisson brackets.[7],

$$(q_i, H) = \dot{q}_i \tag{62.15}$$

$$(p_i, H) = \dot{p}_i \tag{62.16}$$

$$\tag{62.17}$$

Since the value of the Poisson bracket of two functions is independent of the coordinate system used to express the brackets, it becomes obvious that Hamilton's equations of motion are canonical invariants, another analog that will show up in the formulation of quantum mechanics.

The total derivative of a function U(q, p, t) is the Poisson bracket of that function with the Hamiltonian plus the partial derivative with respect to time.

$$\frac{dU}{dt} = \sum_m \left(\frac{\partial U}{\partial q_m} \frac{dq_m}{dt} + \frac{\partial U}{\partial p_m} \frac{dp_m}{dt} \right) + \left(\frac{\partial U}{\partial t} \right) \tag{62.18}$$

$$\frac{dU}{dt} = \sum_m \left(\frac{\partial U}{\partial q_m} \frac{\partial H}{\partial p_m} + \frac{\partial U}{\partial p_m} \frac{\partial H}{\partial q_m} \right) + \left(\frac{\partial U}{\partial t} \right) \tag{62.19}$$

Then it is easily seen that: (a) if $U \equiv H$, $dH/dt = \partial H/\partial t$; (b) if U is a constant of the motion, $(U,H) = 0$; and (c) if U is not an explicit function of the time and $(U,H) = 0$, U is a constant of the motion.

[7]Introduced by the French Mathmetician and Physicist, SimÃĹon-Denis Poisson (1781-1840)

It is often convenient in mechanics to transfer from one set of coordinates (q_m, p_m) to another (Q_m, P_m). If the transformation is canonical, there is some function $K(Q,P,t)$ in the transformed system that will play the role of the Hamiltonian and the equations of motion will be preserved in the new system. If the transformation equations are canonical, there exists some function W that can be used to generate the new variables, Q_m and P_m , and establish the new Hamiltonian, $K(Q,P,t)$. The function W is called Hamilton's characteristic function, and may be used to construct for the case of a single particle described in terms of Cartesian coordinates an equation that is similar to the wave equation in quantum mechanics.

$$H\left(q, \frac{\partial W}{\partial q}\right) + \frac{p^2}{2m} + V = E \tag{62.20}$$

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right] + V = E \tag{62.21}$$

$$(\nabla W)^2 = 2m(E - V) \tag{62.22}$$

Hamilton's principal function S(q,P,t) can be defined as follows.

$$S(q,P,t) = W(q,P) - Et \tag{62.23}$$

This equation may be viewed in configuration space as lines of constant S and constant W moving with time. At t=0, the surfaces of constant S coincide with the surfaces of constant W. The phase velocity of the wave front can be shown to be inversely proportional to the momentum so that the wave fronts speed up as the particle slows down, opposite to the anticipated result that the phase velocity of the wave would equate to the particle velocity. In this case, an analogy to geometrical optics is drawn.

$$u = \frac{E}{p} \tag{62.24}$$

The differential equation for Hamilton's characteristic function, W, bears a striking resemblance to the Eikonal equation for electromagnetic waves.

$$(\nabla L)^2 = n^2 \tag{62.25}$$

Light rays moving perpendicular to wave fronts correspond to particle trajectories perpendicular to surfaces of constant L. In this analogy, W plays the same role as L and n plays the same role as $2m(E - V)$. Classical mechanics is then the

field of geometrical optics in which phenomena, such as interference and diffraction, depending upon wavelength cannot occur. Although there is a duality in classical mechanics between particles and waves, the particle is the senior partner and the wave has no opportunity to display itself.

Just as the successful interpretation of blackbody radiation, the photoelectric effect and scattering of light from electrons required consideration of electromagnetic radiation as being composed of photons, other experiments could only be interpreted by assigning particles a wave characteristic. This is done in the Schrodinger's formulation of wave mechanics in which all the above laws of classical mechanics become evident.

63 Wave Nature of Particles

63.1 Diffraction of X-rays from Crystal Lattice

By the early 1900s, the diffraction of light in the visible spectrum by single-, double- and multiple-slit gratings was a well understood phenomena of physics. In these experiments, slits as small as $1x10^{-6}$ meters were used to diffract light of wavelengths in the region of 5000 angstroms as discussed in section 60.1. Diffraction of microwaves and x-rays with wavelengths in the region of one angstrom required much finer gratings. In 1912 Von Laue suggested that the ordered arrangement of atoms in a crystal lattice might provide a diffraction grating of sufficiently small spacing to observe the diffraction of x-rays. Following his suggestion, the diffraction of x-rays from a rock salt crystal was soon reported by Friedrich and Knipping. The mechanics of x-ray diffraction was worked out in 1913 by W.H. Bragg and his son W. L. Bragg, who related the wavelength of the X-rays λ to the crystal lattice spacing d and angle of diffraction θ. W.L. Bragg and his father W. H. Bragg were awarded the Nobel Prize in Physics in 1915 for this work. The equation defining the maxima in the intensity of the diffracted wave is essentially the same as the equation for Fraunhofer diffraction from a single slit.[8]

$$n\lambda = 2d\sin\theta, \tag{63.1}$$

63.2 Diffraction of Electron Beams

[8] W.L. Bragg, "The Diffraction of Short Electromagnetic Waves by a Crystal", Proceedings of the Cambridge Philosophical Society, 17 (1913), 43-57.

Einstein's equations for the energy $E = h\nu$ and momentum $p = E/c$ of a photon, can be combined with the equation for the phase velocity of an electromagnetic wave $c = \nu\lambda$ to obtain a relationship between the wavelength and the momentum of the photon.

$$\lambda = \frac{h}{p} \tag{63.2}$$

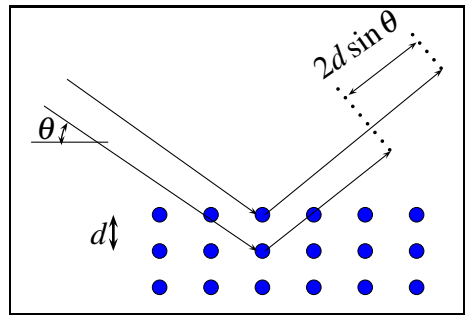


Figure 10.4: Diffraction of light rays of wavelength λ from a crystal with lattice spacing d

Using the accepted value of Plank's constant, $h = 6.625 \times 10^{-34}$ Joule-sec, the mass of the electron, $m = 9.11 \times 10^{-31}$ kg, and the kinetic energy of an electron accelerated through a potential drop of V volts, de Broglie's formula can be written in form particularly useful in electron microscopy.

$$\lambda = \sqrt{\frac{150}{V}} \quad \text{angstroms} \tag{63.3}$$

Using the de Broglie wavelength for the wavelength of an electron Bragg's law, equation 63.1, is still valid for the angle of scattering. In 1924, Louis de Broglie proposed in his doctoral thesis that due to the symmetry of nature material particles might exhibit wave properties and that the equations $E = h\nu$ and $\lambda = h/p$ might apply to all particles. The above equation became known as de Broglie's hypothesis and the wavelength of particles calculated with this equation became known as the de Broglie wavelength.[9]. For his work, Louis de Broglie was awarded the Nobel Prize in Physics in 1929.

Since the de Broglie wavelength of an electron accelerated through a potential of 150 volts is about one Angstrom $(10^{-8}cm)$ comparable to the lattice spacing of many crystals, it was reasonable to expect that diffraction of particles could be observed. These equations can be combined to relate the energy of a particle to its angle of scattering from a diffraction grating to get

$$E = \frac{h^2}{8md^2\sin^2\theta} \tag{63.4}$$

For electrons to undergo first order $(n = 1)$ diffraction in a crystal spectrometer with lattice spacing of about 1 A 10^{-10} meters similar to x-rays, the energy

[9] L. de Broglie, Recherches sur la thÃĹorie des quanta (Researches on the quantum theory), Thesis (Paris), 1924; L. de Broglie, Ann. Phys. (Paris) 3, 22 (1925).

of the electron would have to be around 46 eV. At Bell labs in 1927, Davisson and Germer were studying the angular distribution of electrons scattered from a nickel target when a laboratory accident resulted in crystallization of the target into several large nickel crystals. When the experiment was continued, Davisson and Germer noticed that diffraction maxima occurred in the observed angular distribution that corresponded to maxima that would have been expected in the diffraction of x-rays from a nickel crystal. A 54 eV electron beam and de Broglie wavelength 1.67 angstroms incident upon a single crystal of nickel with a lattice spacing of 0.91 produced a pronounced diffraction peak at $\theta = 65^o$ that could be attributed to first order diffraction using the Bragg Equation.[10] The experiment was soon replicated by G. P. Thomson using much higher energy electrons. Davisson and Thomson were awarded the Nobel Prize in Physics in 1937 and electron diffraction became a well known phenomena of physics.

63.3 Diffraction of Heavy Particles

In 1936 Elsasser and in 1937 Wick suggested that the diffraction of slow neutrons might be observed with crystals. Neutrons, with a mass 2000 times larger than electrons, would have to be thermalized with energies around 0.025 eV to undergo first order diffraction in the same spectrometer. In 1936, Neutron diffraction was first observed by Mitchell and Powers and independently Halban and Prieswerk. Subsequently, it was found that crystaline substances such as Calcium Flouride, Lithum Fluroide and Magnesium Oxide were useful in slow neutron diffraction. Neutron diffraction was used to define the energy of a neutron beam and later is crystallography, also making neutron diffraction a commonly used technique.

The diffraction of particles by crystalline substances is not limited to electrons and neutrons. As early as 1930, Eastermann and Stern used a Lithium Fluoride crystal to diffract Helium atoms with a de Broglie wavelength of 0.6. Thus, in there appeared to be no restriction on the application of the technique as long as diffraction gratings of the proper dimensions can be found, and this made it necessary to recognize that particles had a wave nature that could be utilized to explain physical observations.

[10] C.Davisson, L.H. Germer (1927). "Reflection of electrons by a crystal of nickel". Nature 119: 558-560.

Problems

256. X-rays of wavelength 2.541 A are diffracted from a quartz crystal with lattice spacing of 4.255 A. Calculate the angle between the X-ray beam and the atomic planes for first, second and third order diffraction. ans. 17.37, 36.67, 63.61 degrees

257. Calculate the angle at which a peak should be observed in the electron diffraction pattern for 54 eV electrons scattered from a single crystal of nickel in which the spacing between the Bragg planes is 0.91 A. ans. 66.3 degrees

258. At what angle will thermal neutrons having de Broglie wavelength of 1.80 A be diffracted from a crystal surface if the successive layers of atoms are separated by 1.15 A? ans. 51.5 degrees

259. Show that the de Broglie wavelength of a neutron having a temperature T defined by $K = \frac{1}{2}mv^2 = \frac{3}{2}kT$, where k is Boltzman's constant is given by $\lambda = h/\sqrt{3mkT}$ where m is the mass of the neutron.

64 Uncertainty in Physical Measurements

Heisenberg's uncertainty principle is manifest in many phenomena. One of the most common, as well as first noticed, is diffraction of electrons from a hypothetical diffraction grating with a single slit. In the simplified diagram at right, the momentum of the electron is represented by **p**. Along the x-axis, the momentum $p_x = p\cos\theta$ can be defined with infinite accuracy. However, it is apparent that there is a spread in the momentum of the electron in the y-direction, $\triangle p_y \leq p\sin\theta$. From the diffraction equation $\lambda = 2d\sin\theta$, a substitution of $\lambda/2d$ can be made for $\sin\theta$ and $\triangle y$ for d in the uncertainty of the momentum to obtain.[11]

$$\triangle p_y \leq p\frac{\lambda}{2d} = \frac{h}{\lambda}\frac{\lambda}{2\triangle y} = \frac{h}{2\triangle y} \tag{64.1}$$

which gives upon rearrangement.

$$\triangle y \triangle p_y \leq \frac{h}{2} \tag{64.2}$$

[11] Heisenberg, W. "Ueber den anschaulichen Inhalt der quantentheoretischen Kinematik and Mechanik" Zeitschrift fur Physik 43 172 – 198 (1927)

Thus it is apparent that although there is no restriction on the precision with which x and p_x may be defined, the precision which y and p_y may be defined is limited by the uncertainty principle. The more precise y is defined, the less precise p_y can be known. There is also no restriction on defining x and y or x and p_y. From this it should be apparent that the uncertainty principle is embedded in nature and that it becomes more important as the dimensions of the observed phenomena decrease. At the macroscopic level, the uncertainties become insignificant, but at the atomic level the uncertainty can become comparable in magnitude to the magnitude of the uncertain quantity. This evidence formed the basis for Werner Heisenberg's insight into the uncertainty principle.

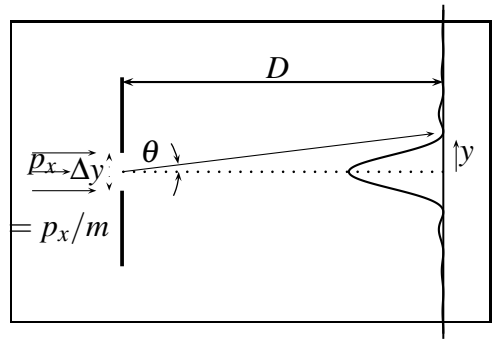


Figure 10.5: Illustration of the relation between slit width and width of the maxima in the diffracted wave

Problems

260. A beam of Silver atoms for a Stern-Gerlach experiment is produced by collimating atoms that vaporize from Silver held in a furnace at 1200 degrees C. Use the uncertainty relation to find the diameter d of the smallest spot that can be obtained at a detector when the Silver atoms pass through a collimator of width Δy and the detector is placed 1.000 meter from the opening in the collimator. ans. 2×10^{-4} cm Hint. First find an inequality for d in terms of Δy and remove Δy by completing the square.

261. If an atom in an excited state in a Frank-Hertz experiment has a lifetime of 1×10^{-10} seconds, use the uncertainty relation between Energy and time to find the uncertainty in the energy level. ans. 3×10^{-4} eV.

65 Correspondence Rule

Another principle which came to guide the development of quantum mechanics is the **correspondence rule**. Stated simply, the correspondence rule says that the rules which describe physical phenomena in microscopic systems such as the atom

will reduce to the rules of classical mechanics which describe physical phenomena in macroscopic systems. Compton scattering is an excellent example of this rule. If, for example, electromagnetic radiation with wavelength of 1 A, comparable to the diameter of the first Bohr orbit in the hydrogen atom, undergoes Compton scattering at 90 degrees with an electron at rest, the shift in wavelength predicted by equation 61.20(a) will be 0.02426 A, equivalent to 0.02426/1=2.426% of the incident wavelength. In this case the incident radiation looses over 2% of its energy to the scattering electron

If, on the other hand, an X-ray with wavelength in the visible range, 4000 A, undergoes Compton scattering at 90 degrees with an electron, the change in wavelength predicted by equation 61.20(a) will still be only 0.02426 A, equivalent to 0.02426/4000=0.00061%. Thus the wavelength, or frequency, of the incident and scattered radiation is essentially the same as expected in classical mechanics and the correspondence rule holds true.

66 Quantization of Physical Systems

As the emission spectra of radiation from atoms was examined in the early part of the 20th century, it was found that energy was emitted only in discrete amounts and not in a continuous spectra. This could only be explained if it was assumed that the energy levels of atoms and nuclei were restricted to discrete values that depended on the specific atom or nucleus and level of excitation.

As one example of the quantization of physical systems, we will examine the Bohr theory of atomic structure, which was introduced in 1913 by Neils Bohr (1885-1962) a Danish physicist who received the Nobel Prize in Physics in 1922. At the time Bohr introduced his theory of the hydrogen atom in 1913 the quantum nature of radiation was well established but the wave nature of particles had not been introduced. Bohr took the atom to be a planetary system of electrons rotating about a much larger nucleus and based his theory on four postulates:

- An atomic system possesses a number of stationary states from which no emission takes place.
- An emission of radiation results from a transition from one state to another.
- The possible stationary states of an atomic system is determined by quantization of the angular momentum.

$$p = mvr = n\hbar \quad \text{where } n = 1, 2, 3, \cdots \tag{66.1}$$

- The dynamics of stationary states are controlled by the laws of classical mechanics so that in any stationary state the centripetal force on an orbital electron is balanced by the electrostatic force of attraction to the nucleus.

$$\frac{mv^2}{r} = \frac{Ze^2}{4\pi\varepsilon_o r^2} \tag{66.2}$$

Working backwards from the fourth postulate and employing the third we obtain a quantization of velocity and radius.

$$v = \frac{Ze^2}{2\varepsilon_o hn} \tag{66.3a}$$

$$r = \frac{\varepsilon_o h^2 n^2}{m\pi e^2 Z} \tag{66.3b}$$

Taking the quantum number of the orbit to be the value of n, the radius of the first orbit of the Hydrogen atom is seen to be

$$r_1 = \frac{\varepsilon_o h^2}{m\pi e^2 Z} = 5.29 \times 10^{-9}\text{cm}. \tag{66.4}$$

which became known as the Bohr radius. The velocity of the electron in the first orbit was also found to be

$$v_1 = \frac{Ze^2}{\varepsilon_o h} = 2.2 \times 10^8\text{cm/sec}. \tag{66.5}$$

The energy of the orbital electron was then obtained by taking the potential energy of the electron to be the electrostatic energy and using the fourth postulate to obtain a similar expression for the kinetic energy.

$$U = -\frac{Ze^2}{4\pi\varepsilon_o r} \tag{66.6a}$$

$$K = \frac{1}{2}mv^2 = \frac{Ze^2}{8\pi\varepsilon_o r} \tag{66.6b}$$

so that the quantized energy of the orbital electron obtained from the sum of the potential and kinetic energies becomes.

$$E_n = -\frac{Ze^2}{8\pi\varepsilon_o r} = \frac{me^4 Z^2}{8\varepsilon_o^2 h^2}\frac{1}{n^2} \tag{66.7}$$

Upon replacing m with the reduced mass of the proton and orbital electron and setting Z=1, the energies of the Bohr orbits allowed for the hydrogen atoms are easily calculated to be -13.57, -3.39, -1.51, -0.85, etc for n=1, 2, 3, and 4. Bohr's second postulate allows us to say that as the electron in the hydrogen atom jumps from one state to another, energy is emitted in the form of a single quantum with frequency

$$\nu = \frac{me^4}{8\varepsilon_o^2 h^3}\left(\frac{1}{n_i^2} - \frac{1}{n_j^2}\right) \tag{66.8}$$

Thus it became apparent that a series of radiation quanta would be emitted as electrons transition from higher states to any one state with a lower value of n as illustrated in figure 10.6

At the time Bohr proposed his theory, only the Balmer and Paschen series for hydrogen were known. One of the amazing successes of Bohr's theory was that not only were the frequencies of the spectral lines of the Balmer and Paschen series were correctly predicted but the existence of three heretofore unknown series was predicted. The Lyman series was found in 1916, the Brackett series was found in 1922, and the Pfund series was found in 1924 and the frequencies of all spectral lines agreed with the predictions of Bohr's theory.

Problems

262. Compute the energies of the first four levels of hydrogen. ans. 13.57, 3.39, 1.51, 0.848 eV

263. Compute the frequencies of transitions from the second, third and fourth levels to the first. 1,218, 1,028, 975 A

264. Compute the waelength of each wave. 2.461×10^{15}, 2.917×10^{15}, 3.076×10^{15} Hertz.

67 Summary Development of Quantum Mechanics

It was clear from these paradoxes that classical mechanics and the known physics could not explain the behavior of atomic systems and the search began in earnest during the 1920s for a way that these naturally observed phenomena could be explained.

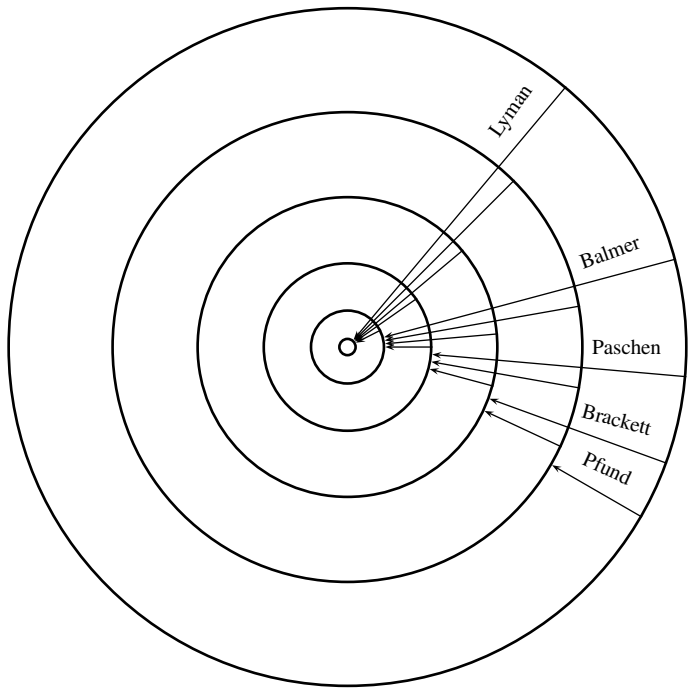


Figure 10.6: Spectral lines of the hydrogen atom in the Bohr theory.

Following the developments of the early part of the twentieth century, which showed that an electron could exhibit both particle and wave characteristics, it became clear that classical mechanics could not be modified to give an accurate and complete explanation of atomic phenomena. Invoking quantization rules did not allow full and complete explanations. Therefore, a new formulation of mechanics was required which would describe both wave and particle characteristics within the same framework. Unfortunately, there existed no logical sequence of mathematical steps, which could lead to this new formulation. As a result it was necessary to resort to intuition and base the proof on the results. Two basic formulations were set forth between the years 1923 and 1927. These formulations became know as matrix mechanics and wave mechanics.

In his first paper on quantum mechanics, Heisenberg developed a formal method for calculating values of the frequencies and intensities of the spectral lines, which an atomic system could emit or absorb. Heisenberg took as his starting point a critical analysis of the old quantum theory and assumed that variables, which were observable, could be distinguished from those, which were not. This idea led Heisenberg to ignore the concept of an electron orbit in the Hydrogen atom and associate a matrix with each physical observable. Contrary to the laws of modern algebra, these matrices did not commute, and equations of motion of the system became equations between matrices. It was soon noted by Born and Jordan that Heisenberg's mechanics was essentially matrix algebra already in use by mathematicians. Heisenberg's mechanics became known as matrix mechanics, development was rapidly completed and the new mechanics was applied to numerous problems.

Erwin SchrËĘdinger took the opposite view and adopted de Broglie's thesis that there was an intrinsic wave associated with each particle. Working with the idea of matter waves, SchrËĘdinger succeeded in the development of a wave equation, which these matter waves were supposed to satisfy. This equation became the basic element of wave mechanics. Later SchrËĘdinger and Echart rectified the apparent discrepancy between the two forms of mechanics when he showed that they were equivalent formulations. Development of the new mechanics proceeded rapidly. One of the most difficult problems was reconciliation of the new mechanics with the requirements of general relativity. P.A.M. Dirac solved this problem in a more general formalism, which encompassed both matrix and wave mechanics as particular formulations. The generally accepted way of introducing quantum mechanics is by way of the wave formulation due to SchrËĘdinger. Wave mechanics lends itself more readily to physical interpretation than matrix mechanics and will provide the background for development of matrix mechanics.

Appendix A

Fundamental Constants

Quantity	*Symbol*	*Value*	*Unit*	*Uncertainty*
Speed Light (vacumn)	c	299792458	m/sec	
magnetic moment	μ_o	$4\pi \times 10^{-7}$	N/A^2	
electric constant	ε_o	$8.854187817 \times 10^{-12}$	N/A^2	
gravitational constant	G	$6.67428(67) \times 10^{-11}$	$N\,m^2/kg^2$	
Plank constant	h	$6.62606896(33) \times 10^{-34}$	$J\,sec$	
Plank constant/2π		$1.054571628(53) \times 10^{-34}$	$J\,sec$	
electron charge	e	$1.602176487(40) \times 10^{-19}$	C	
electron mass	m_e	$9.10938215(45) \times 10^{-31}$	kg	
proton mass	m_p	$1.672621637(83) \times 10^{-27}$	kg	
fine-structure constant	α	$7.2973525376(50) \times 10^{-3}$		
	$1/\alpha$	137.035999679(94)		
Rydberg constant	R_∞	10973731.568527(73)	m^{-1}	
Avagadro's number	N_o	$6.02214179(30) \times 10^{23}$	$mole^{-1}$	
molar gas constant	R	8.314472(15)	$J/mole/^oK$	
Boltzmann constant	k	$1.3806504(24) \times 10^{-23}$	$J/^oK$	
Stefan-Boltzmann constant	σ	$5.670400(40) \times 10^{-8}$	$W/m^2\,K^{-4}$	

Appendix B

Conversion Factors

Quantity	*MKS*		*CGS*		*BRITISH*	
Acceleration	1	m/s^2	100	cm/s^2	3.281	ft/s^2
Area	1	m^2	10^4	cm^2	10.764	ft^2
Density	1	kg/m^3	0.001	g/cm^3	0.06243	lbs/ft^3
Energy	1	$N\,m$	10^7	$dyne\ cm$	0.7376	$ft\ lb$
Energy	1	$Joule$	10^7	$ergs$	0.0009481	Btu
Energy	1	$Joule$	10^7	$ergs$	0.2389	$Calories$
Force	1	N	10^5	$dynes$	0.2248	lb
Heat Transfer Coefficient	1	$Wm^2/^oK$	$2.38846E-05$	$cal/cm^2\ s\ ^oC$	0.1761	$Btu/ft^2\ h\ ^oF$
Specific Heat Capacity	1	$J/kg\ ^oK$	10^4	$ergs/g/^oC$	0.000238846	$Btu/lb\ ^oF$
Length	1	m	100	cm	3.281	ft
Mass	1	kg/m^3	1000	g	0.06852	$slugs$
Moment of Inertia	1	$kg\ m^2$	10000	$kg\ cm^2$	23.73	$lb\ ft^2$
Power	1	$Joules/sec$	10^7	$ergs/sec$	0.7376	$ft\ lb/sec$
Power	1	$watt$	10^7	$ergs/sec$	3.41316	Btu/hr
Power	1	$watt$	10^7	$ergs/sec$	0.2389	$Calories/sec$
Pressure	1	Pa	10	$dynes/cm^2$	0.00014504	lb/in^2
Thermal Conductivity	1	$watt/m^oK$	$2.3885E-03$	$cal/cm\ s\ ^oC$	0.5779	$Btu/ft\ h^oF$
Velocity	1	m/s	100	cm/s	3.281	ft/s
Viscosity	1	$Poise$	1	$gm/cm\ sec$	0.06721	$lb/ft\ sec$

1 $cal/g/^oC$ = 1 $kcal/kg/^oC$ = 1 $Btu/lb/^oF$ = 1 $Btu/lb/^oR$

F = (9/5)C + 32 $\Delta F = \Delta R = 1.8\Delta K$

C = (5/9)F - 32 $\Delta C = \Delta K = 0.5555\Delta F$

K = C+ 273.15 $\Delta K = \Delta C = 0.5555\Delta KF$

R = F + 459.67 $\Delta R = \Delta F = 1.8\Delta K$

1 liter-atm = 24.20057 calories

Appendix C

SI Derived Quantities

SI Derived Quantity	Name	*Symbol*	*SI Derived Unit*	*SI Fundamental Unit*
area	square meter			m^2
volume	cubic meter			m^3
speed, velocity	meter per second			m/s
acceleration	meter per second squaredâĂăâĂă			m/s^2
wave number	reciprocal meter			m^{-1}
mass density	kilogram per cubic meter			kg/m^3
specific volume	cubic meter per kilogram			m^3/kg
current density	ampere per square meter			A/m^2
magnetic field strengthâĂăâĂă	ampere per meter			A/m
amount-of-substance concentration	mole per cubic meter			mol/m^3
luminance	candela per square meter			cd/m^2
mass fraction	kilogram per kilogram			$kg/kg = 1$
plane angle	radian (a)	rad	$-$	$m\,m^{-1} = 1(b)$
solid angle	steradian (a)	$sr(c)$	$-$	$m^2\,m^{-2} = 1(b)$
frequency	hertz	Hz	$-$	s^{-1}
force	newton	N	$-$	$m\,kg\,s^{-2}$
pressure, stress	pascal	Pa	N/m^2	$m^{-1}\,kg\,s^{-2}$
energy, work, quantity of heatâĂăâĂă	joule	J	$N\,m$	$m^2\,kg\,s^{-2}$
power, radiant flux	watt	W	J/s	$m^2\,kg\,s^{-3}$
electric charge, quantity of electricity	coulomb	C	$-$	$s\,A$
electric potential difference, electromotive force	volt	V	W/A	$m^2\,kg\,s^{-3}\,A^{-1}$
capacitance	farad	F	C/V	$m^{-2}\,kg^{-1}\,s^4\,A^2$
electric resistance	ohm		V/A	$m^2\,kg\,s^{-3}\,A^{-2}$
electric conductance	siemens	S	A/V	$m^{-2}\,kg^{-1}\,s^3\,A^2$
magnetic flux	weber	Wb	Vs	$m^2\,kg\,s^{-2}\,A^{-1}$
magnetic flux density	tesla	T	Wb/m^2	$kg\,s^{-2}\,A^{-1}$
inductance	henry	H	Wb/A	$m^2\,kg\,s^{-2}\,A^{-2}$

SI Derived Quantities continued

			SI Derived	*SI Fundamental*
Celsius temperature	degree Celsius	C	$-$	K
luminous flux	lumen	lm	$cdsr(c)$	$m^2\ m^{-2}\ cd = cd$
illuminance	lux	lx	lm/m^2	$m^2\ m^{-4}\ cd = m^{-2}\ cd$
activity (of a radionuclide)	becquerel	Bq	$-$	s^{-1}
absorbed dose	gray	Gy	J/kg	$m^2\ s^{-2}$
dose equivalent (d)	sievert	Sv	J/kg	$m^2\ s^{-2}$
catalytic activity	katal	kat		$s^{-1}\ mol$
dynamic viscosity	pascal second		$Pa\ s$	
moment of force	newton meter		$N\ m$	
surface tension	newton per meter		N/m	
angular velocity	radian per second		rad/s	
angular acceleration	radian per second squared		rad/s^2	
heat flux density, irradiance	watt per square meter		W/m^2	
heat capacity, entropy	joule per kelvin		J/K	
specific heat capacity, specific entropy	joule per kilogram kelvin		$J/(kgK)$	
specific energy	joule per kilogram		J/kg	
thermal conductivity	watt per meter kelvin		$W/(mK)$	
energy density	joule per cubic meter		J/m^3	
electric field strength	volt per meter		V/m	
electric charge density	coulomb per cubic meter		C/m^3	
electric flux density	coulomb per square meter		C/m^2	
permittivity	farad per meter		F/m	
permeability	henry per meter		H/m	
molar energy	joule per mole		J/mol	
molar entropy, molar heat capacity	joule per mole kelvin		$J/(molK)$	
exposure (x and rays)	coulomb per kilogram		C/kg	
absorbed dose rate	gray per second		Gy/s	
radiant intensity	watt per steradian		W/sr	
radiance	watt per square meter steradian		$W/(m^2sr)$	
catalytic (activity) concentration	katal per cubic meter		kat/m^3	

[1]

[1] By convention the first letter of its symbol for all units whose names are derived from the proper name of a person is uppercase, but when the unit is spelled out, it should always be written in lowercase, unless it begins a sentence.

Appendix D

Solar System Data

Characteristic	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
$MeanDistancefromtheSun(AU)$	0.3871	0.7233	1	1.524	5.203	9.539	19.19	30.06	39.48
$Siderealperiodoforbit(years)$	0.24	0.62	1	1.88	11.86	29.46	84.01	164.79	248.54
$MeanOrbitalVelocity(km/sec)$	47.89	35.04	29.79	24.14	13.06	9.64	6.81	5.43	4.74
$OrbitalEccentricity$	0.206	0.007	0.017	0.093	0.048	0.056	0.046	0.01	0.248
$Inclinationtoecliptic(degrees)$	7	3.4	0	1.85	1.3	2.49	0.77	1.77	17.15
$EquatorialRadius(km)$	2439	6052	6378	3397	71490	60268	25559	25269	1160
$PolarRadius(km)$	same	same	6357	3380	66854	54360	24973	24340	same
$Massofplanet(Earth=1)$	0.06	0.82	1	0.11	317.89	95.18	14.53	17.14	0.002
$Meandensity(grams/centimeter^3)$	5.43	5.25	5.52	3.95	1.33	0.69	1.29	1.64	2.03
$Bodyrotationperiod(hours)$	1408	5832	23.93	24.62	9.92	10.66	17.24	16.11	153.3
$Tiltofequatortoorbit(degrees)$	2	177.3	23.45	25.19	3.12	26.73	97.86	29.6	122.46
$Numberofobservedsatellites$	0	0	1	2	28	30	24	8	1
$Albedo$	0.106	0.65	0.367	0.15	0.52	0.47	0.51	0.41	0.3
$Equatorialgravity(m/sec^2$	3.7	8.87	9.8	3.71	24.79	10.44	8.87	11.15	0.66
$Escapevelocity(m/sec)$	4.25	10.36	11.19	5.03	60.2	36.09	21.38	23.56	1.23

National Air and Space Museum, 2002
Mass of Earth = 5.972190x10^{24} kg

Appendix E

Properties of the Elements

Z	Element	Young's Modulus Coeff Y Gpa	Shear Modulus Coeff S Gpa	Bulk Modulus Coeff β Gpa	Thermal Expansion Coeff α K^{-1}	Thermal Conductivity Coeff k W/(m K)	Specific Heat J/(kg K)
89	Actinium	N/A	N/A	N/A	N/A	12	120
13	Aluminum	70	26	76	0.0000231	235	904
95	Americium	N/A	N/A	N/A	N/A	10	N/A
51	Antimony	55	20	42	0.000011	24	207
18	Argon	N/A	N/A	N/A	N/A	0.01772	520.33
33	Arsenic	8	N/A	22	N/A	50	328
85	Astatine	N/A	N/A	N/A	N/A	2	N/A
56	Barium	13	4.9	9.6	0.0000206	18	205
97	Berkelium	N/A	N/A	N/A	N/A	10	N/A
4	Beryllium	287	132	130	0.0000113	190	1820
83	Bismuth	32	12	31	0.0000134	8	122
107	Bohrium	N/A	N/A	N/A	N/A	N/A	N/A
5	Boron	N/A	N/A	320	0.000006	27	1030
35	Bromine	N/A	N/A	1.9	N/A	0.12	947.3
48	Cadmium	50	19	42	0.0000308	97	230
20	Calcium	20	7.4	17	0.0000223	200	631
98	Californium	N/A	N/A	N/A	N/A	N/A	N/A
6	Carbon	N/A	N/A	33	0.0000071	140	710
58	Cerium	34	14	22	0.0000063	11	192
55	Cesium	1.7	N/A	1.6	N/A	36	242
17	Chlorine	N/A	N/A	1.1	N/A	0.0089	478.2
24	Chromium	279	115	160	0.0000049	94	448
27	Cobalt	209	75	180	0.000013	100	421
29	Copper	130	48	140	0.0000165	400	384.4
96	Curium	N/A	N/A	N/A	N/A	N/A	N/A
110	Darmstadtium	N/A	N/A	N/A	N/A	N/A	N/A
105	Dubnium	N/A	N/A	N/A	N/A	N/A	N/A
66	Dysprosium	61	25	41	0.0000099	11	167
99	Einsteinium	N/A	N/A	N/A	N/A	N/A	N/A
68	Erbium	70	28	44	0.0000122	15	168
63	Europium	18	7.9	8.3	0.000035	14	182

Z	Element	Young's Modulus Coeff Y Gpa	Shear Modulus Coeff S Gpa	Bulk Modulus Coeff β Gpa	Thermal Expansion Coeff α K^{-1}	Thermal Conductivity Coeff k W/(m K)	Specific Heat J/(kg K)
100	Fermium	N/A	N/A	N/A	N/A	N/A	N/A
9	Fluorine	N/A	N/A	N/A	N/A	0.0277	824
87	Francium	N/A	N/A	N/A	N/A	N/A	N/A
64	Gadolinium	55	22	38	0.0000094	11	240
31	Gallium	N/A	N/A	N/A	0.00012	29	371
32	Germanium	N/A	N/A	N/A	0.000006	60	321.4
79	Gold	78	27	220	0.0000142	320	129.1
72	Hafnium	78	30	110	0.0000059	23	144
108	Hassium	N/A	N/A	N/A	N/A	N/A	N/A
2	Helium	N/A	N/A	N/A	N/A	0.1513	5193.1
67	Holmium	65	26	40	0.0000112	16	165
1	Hydrogen	N/A	N/A	N/A	N/A	0.1805	14300
49	Indium	11	N/A	N/A	0.0000321	82	233
53	Iodine	N/A	N/A	7.7	N/A	0.449	429
77	Iridium	528	210	320	0.0000064	150	131
26	Iron	211	82	170	0.0000118	80	449
36	Krypton	N/A	N/A	N/A	N/A	0.00943	248.05
57	Lanthanum	37	14	28	0.0000121	13	195
103	Lawrencium	N/A	N/A	N/A	N/A	N/A	N/A
82	Lead	16	5.6	46	0.0000289	35	127
3	Lithium	4.9	4.2	11	0.000046	85	3570
71	Lutetium	69	27	48	0.0000099	16	154
12	Magnesium	45	17	45	0.0000082	160	1020
25	Manganese	198	N/A	120	0.0000217	7.8	479
109	Meitnerium	N/A	N/A	N/A	N/A	N/A	N/A
101	Mendelevium	N/A	N/A	N/A	N/A	N/A	N/A
80	Mercury	N/A	N/A	25	N/A	8.3	139.5
42	Molybdenum	329	20	230	0.0000048	139	251
60	Neodymium	41	16	32	0.0000096	17	190
10	Neon	N/A	N/A	N/A	N/A	0.0491	1030
93	Neptunium	N/A	N/A	N/A	N/A	6	N/A
28	Nickel	200	76	180	0.0000134	91	445
41	Niobium	105	38	170	0.0000073	54	265
7	Nitrogen	N/A	N/A	N/A	N/A	0.02583	1040
102	Nobelium	N/A	N/A	N/A	N/A	N/A	N/A
76	Osmium	N/A	222	N/A	0.0000051	88	130
8	Oxygen	N/A	N/A	N/A	N/A	0.02658	919
46	Palladium	121	44	180	0.0000118	72	240
15	Phosphorus	N/A	N/A	11	N/A	0.236	769.7
78	Platinum	168	61	230	0.0000088	72	133
94	Plutonium	96	43	N/A	N/A	6	N/A
84	Polonium	N/A	N/A	N/A	N/A	N/A	N/A
19	Potassium	N/A	1.3	3.1	N/A	100	757

Z	Element	Young's Modulus Coeff Y GPa	Shear Modulus Coeff S GPa	Bulk Modulus Coeff β GPa	Thermal Expansion Coeff α K^{-1}	Thermal Conductivity Coeff k W/(m K)	Specific Heat J/(kg K)
59	Praseodymium	37	15	29	0.0000067	13	193
61	Promethium	46	18	33	0.000011	15	N/A
91	Protactinium	N/A	N/A	N/A	N/A	47	99.1
88	Radium	N/A	N/A	N/A	N/A	19	92
86	Radon	N/A	N/A	N/A	N/A	0.00361	93.65
75	Rhenium	463	178	370	0.0000062	48	137
45	Rhodium	275	150	380	0.0000082	150	240
111	Roentgenium	N/A	N/A	N/A	N/A	N/A	N/A
37	Rubidium	2.4	N/A	2.5	N/A	58	364
44	Ruthenium	447	173	220	0.0000064	120	238
104	Rutherfordium	N/A	N/A	N/A	N/A	N/A	N/A
62	Samarium	50	20	38	0.0000127	13	196
21	Scandium	74	29	57	0.0000102	16	567
106	Seaborgium	N/A	N/A	N/A	N/A	N/A	N/A
34	Selenium	10	3.7	8.3	N/A	0.52	321.2
14	Silicon	47	N/A	100	0.0000026	150	710
47	Silver	83	30	100	0.0000189	430	235
11	Sodium	10	3.3	6.3	0.000071	140	1230
38	Strontium	N/A	6.1	N/A	0.0000225	35	300
16	Sulfur	N/A	N/A	7.7	N/A	0.205	705
73	Tantalum	186	69	200	0.0000063	57	140
43	Technetium	N/A	N/A	N/A	N/A	51	63
52	Tellurium	43	16	65	N/A	3	201
65	Terbium	56	22	38.7	0.0000103	11	182
81	Thallium	8	2.8	43	0.0000299	46	129
90	Thorium	79	31	54	0.000011	54	118
69	Thulium	74	31	45	0.0000133	17	160
50	Tin	50	18	58	0.000022	67	217
22	Titanium	116	44	110	0.0000086	22	520
74	Tungsten	411	161	310	0.0000045	170	132
92	Uranium	208	111	100	0.0000139	27	116
23	Vanadium	128	47	160	0.0000084	31	489
54	Xenon	N/A	N/A	N/A	N/A	0.00565	158.32
70	Ytterbium	24	9.9	31	0.0000263	39	154
39	Yttrium	64	26	41	0.0000106	17	298
30	Zinc	108	43	70	0.0000302	120	388
40	Zirconium	68	33	N/A	0.0000057	23	278

Appendix F

Mechanical Properties of Common Materials

Material	Young's Modulus Coeff Y GPa	Shear Modulus Coeff S GPa	Bulk Modulus Coeff β GPa	Thermal Expansion Coeff α K^{-1}	Thermal Conductivity Coeff k W/(m K)	Specific Heat J/(kg K)
Porcelain	104			5.0-6.5	5	
Al_2O_3	400			8.8	30.1	775
optical glass	50-130	19		0.000004	0.5-1.4	
window glass	50-90				0.96	
soda-lime glass				0.000009	1.7	700
AISI 12L14 Steel	200	80	140	0.0000115	51.9	472
304 Stainless	193	78		0.0000173	16.2	502
6061 Aluminum	69	24		0.0000073	167	921
cast iron	83-170	41		0.0000033	80	
structural steel, ASTM A-36	200	79				
polyethylene				60-220	0.38	2100
Polystyrene	3.0-3.5			50-85	0.13	1360
Polypropylene	1.5-2.0					
rubber	0.01-0.1					
brick					0.69	
dry earth					1.5	
Concrete	14	21			1.7	
water					0.6	
ice					2	
plywood		0.62			0.13	
hardwood	11				0.14	
softwood	9				0.12	

Appendix G

Heat Conductance of Common Materials

Gases	k	Metals	k	Liquids	k	Building Materials	k
Air	0.024	Aluminum	250	Alcohol	0.17	Asbestos mill board	0.14
Carbon Dioxide	0.0146	Beryllium	218	Benzene	0.16	Asphalt	0.75
Oxygen	0.024	Cadmium	92	Ethylene glycol	0.25	Brick work	0.69
Nitrogen	00.024	Carbon	1.7	Freon liquid	0.07	Clay, dry to moist	0.15-1.8
Hydrogen	0.168	Copper	401	Kerosene	0.15	Clay, saturated	0.6-2.5
Helium	0.142	Carbon Steel	43	Olive oil	0.17	Concrete, light	0.42
Freon 12	0.073	Gold	310	Water	0.58	Cork	0.07
Argon	0.024	Iridium	147	Engine oil	0.15	Cotton	0.03
Methane	0.030	Lead	35	Gasoline	0.15	Earth, dry	1.5
		Mercury	8			Brick, fireclay	1.4
		Molybdenum	138			Glass	1.05
		Nickel	91			Hardwoods	0.16
		Platinum	70			Paper	0.05
		Silver	429			Plywood	0.13
		Stainless Steel	16			Polyethylene HD	0.42-0.51
		Tin	67			Polyurethane foam	0.02
		Zinc	116			Porous volcanic rock	0.5-2.5
		Brass	109			Sand, dry	0.15-0.25
		Sodium	84			Sand, wet	2-4
						Sawdust	0.08
						Snow	0.05-0.25
						Softwood	0.12
						Sand, wet	2-4
						Soil, organic	0.15-2
						Soil, wet	0.6-4
						Styrofoam	0.033

Table G.1: Thermal conductivities in $Watts/meter/deg\,K$ for common substances at 20 $degrees\,C$.

Appendix H

Electromagnetic Units in SI system

Name of Quantity	Derived Units	*Symbol*	*Unit*	*BaseUnits*
Electric current	ampere (SI base unit)	I	A	$A(=W/V=C/s)$
Electric charge	coulomb	Q	C	As
Potential difference; Electromotive force	volt	V	V	$J/C=kgm^2s^3A^{-1}$
Electric resistance; Impedance; Reactance	ohm	R	Ω	$V/A=kgm^2s^{-3}A^{-2}$
Electric reactance	ohm	X	Ω	$V/A=kgm^2s^{-3}A^{-2}$
Electric impedance	ohm	Z	Ω	$V/A=kgm^2s^{-3}A^{-2}$
Resistivity	ohm metre	ρ	ohm^{-1}	$kgm^3s^{-3}A^{-2}$
Electric power	watt	P	W	$VA=kgm2s?3$
Capacitance	farad	C	F	$C/V=kg^{-1}m^{-2}A^2s^4$
Electric field strength	volt/metre	E	V/m	$N/C=kgmA^{-1}s^{-3}$
Electric displacement field	Coulomb/square metre	D	C/m^2	Asm^{-2}
Permittivity	farad per metre	ε	F/m	$kg^{-1}m^{-3}A^2s^4$
Electric susceptibility	(dimensionless)	Ξ_e		
Conductance	siemens	G	S	$\Omega^{-1}=kg^{-1}m^{-2}s^3A^2$
Admittance	siemens	Y	Y	$\Omega^{-1}=kg^{-1}m^{-2}s^3A^2$
Susceptance	siemens	B	B	$\Omega^{-1}=kg^{-1}m^{-2}s^3A^2$
Conductivity	siemens per metre	k	S/m	$kg^{-1}m^{-3}s^3A^2$
Magnetic flux density, Magnetic induction	tesla	B	T	$Wb/m^2=kgs^{-2}A^{-1}=NA^{-1}m^{-1}$
Magnetic flux	weber	Φ	Wb	$Vs=kgm^2s^{-2}A^{-1}$
Magnetic field strength	ampere/metre	H	A/m	Am^{-1}
Self Inductance	henry	L	H	$Wb/A=Vs/A=kgm^2s^{-2}A^{-2}$
Mutual Inductance	henry	M	H	$Wb/A=Vs/A=kgm^2s^{-2}A^{-2}$
Permeability	henry per metre	μ	H/m	$kgms^{-2}A^{-2}$
Magnetic susceptibility	(dimensionless)	Ξ		

Table H.1: Electromagnetic Units

Appendix I

Table of Prefixes

Prefix	Symbol	Value
Pico	p	10^{-12}
Nano	n	10^{-9}
Micro	μ	10^{-6}
Milli	m	10^{-3}
Centi	c	10^{-2}
Deci	d	10^{-1}
Deka	D	10^{+1}
Hecto	H	10^{+2}
Kilo	K	10^{+3}
Mega	M	10^{+6}
Giga	G	10^{+9}
Tara	T	10^{+12}

Table I.1: Schedule of prefixes