Time-ordering vs normal-ordering and the two-point function/propagator

[+8] [1] Dilaton

[2011-12-09 23:04:53]

[quantum-mechanics quantum-field-theory conformal-field-theory wick-theorem]

[https://physics.stackexchange.com/questions/18078/time-ordering-vs-normal-ordering-and-the-two-point-function-propagator]

I don't understand how to calculate this generalized two-point function or propagator, used in some advanced topics in quantum field theory, a normal ordered product (denoted between ::) is subtracted from the usual time ordered product (denoted T):

$$\langle X^\mu(\sigma,\tau)X^\nu(\sigma',\tau')\rangle \;=\; T(X^\mu(\sigma,\tau)X^\nu(\sigma',\tau')) \;-\; :X^\mu(\sigma,\tau)X^\nu(\sigma',\tau'):$$

My question is can the rhs of this propagator be derived or the meaning of the subtraction of the time ordered product explained and motivated in simple words?

[+15] [2011-12-10 01:25:25] Qmechanic [✔ACCEPTED]

If the operators X_i can be written as a sum of an annihilation and a creation part[1]

$$X_i \;=\; A_i + A_i^\dagger, \qquad i \in I, \tag{1}$$

$$A_i|0\rangle \;=\; 0, \qquad \langle 0|A_i^\dagger \;=\; 0, \qquad i \in I, \tag{2}$$

where

$$[A_i(t), A_j(t')] \;=\; 0, \qquad [A_i^\dagger(t), A_j^\dagger(t')] \;=\; 0, \qquad i,j \in I, \tag{3}$$

and

$$[A_i(t), A_j^\dagger(t')] \;=\; (c\text{ number}) \times \mathbf{1}, \qquad i,j \in I, \tag{4}$$

i.e. proportional to the identity operator $\mathbf{1}$, then one may prove that

$$T(X_i(t)X_j(t')) \;-\; :X_i(t)X_j(t'): \;=\; \langle 0|T(X_i(t)X_j(t'))|0\rangle\,\mathbf{1}. \tag{5}$$

Proof of eq. (5): On one hand, the time ordering T is defined as

$$\begin{aligned} T(X_i(t)X_j(t')) &= \Theta(t-t')X_i(t)X_j(t') + \Theta(t'-t)X_j(t')X_i(t) \\ &= X_i(t)X_j(t') - \Theta(t'-t)[X_i(t),X_j(t')] \\ &\overset{(1)+(3)}{=} X_i(t)X_j(t') - \Theta(t'-t)\left([A_i(t),A_j^\dagger(t')] + [A_i^\dagger(t),A_j(t')]\right). \end{aligned} \tag{6}$$

On the other hand, the normal ordering :: moves by defintion the creation part to the left of the annihilation part, so that

$$:X_i(t)X_j(t'): \;\overset{(1)}{=}\; X_i(t)X_j(t') - [A_i(t),A_j^\dagger(t')], \tag{7}$$

$$\langle 0|:X_i(t)X_j(t'):|0\rangle \;\overset{(1)+(2)}{=}\; 0. \tag{8}$$

The difference of eqs. (6) and (7) is the lhs. of eq. (5):

$$T(X_i(t)X_j(t')) \;-\; :X_i(t)X_j(t'): \stackrel{(6)+(7)}{=} \Theta(t-t')[A_i(t),A_j^\dagger(t')] + \Theta(t'-t)[A_j(t'),A_i^\dagger(t)], \tag{9}$$

which is proportional to the identity operator $\mathbf{1}$ by assumption (4). Now sandwich eq. (9) between the bra $\langle 0|$ and the ket $|0\rangle$. Since the rhs. of eq. (9) is proportional to the identity operator $\mathbf{1}$, the unsandwiched rhs. must be equal to the sandwiched rhs. times the identity operator $\mathbf{1}$. Hence also the unsandwiched lhs. of eq. (9) must also be equal to the sandwiched lhs. times the identity operator $\mathbf{1}$. This yields eq. (5).

--

[1] The operators A_i and $A_i^\dagger$ need not be Hermitian conjugates in what follows.

(1) Thanks a lot @Qmechanic, this clear proof is exactly what I needed. The only thing left to do for me now is to check that the preconditions are valid in my specific case. - **Dilaton**
@Qmechanic, for the operators A and B, is the equation $\langle 0|A+B|0\rangle = \langle 0|A|0\rangle + \langle 0|B|0\rangle$ satisfied? If it is satisfied, then we can sandwich the lhs. of eq. (1) and the equation simply becomes $\langle 0|:X_i(t)X_j(t'):|0\rangle = 0$. But such equation sometimes fails. - **Wein Eld**
I updated the answer. It seems you are talking about situations where one uses definitions of Fock vacuum and normal order that are not properly compatible/adjusted to each other. - **Qmechanic**
1