Time-ordering vs normal-ordering and the two-point function/propagator

I don't understand how to calculate this generalized two-point function or propagator, used in some advanced topics in quantum field theory, a normal ordered product (denoted between ::) is subtracted from the usual time ordered product (denoted $T$):

$$\langle X^\mu(\sigma, \tau)X^\nu(\sigma', \tau') \rangle = T(X^\mu(\sigma, \tau)X^\nu(\sigma', \tau')) - X^\mu(\sigma, \tau)X^\nu(\sigma', \tau')$$

My question is can the rhs of this propagator be derived or the meaning of the subtraction of the time ordered product explained and motivated in simple words?

Proof of eq. (5): On one hand, the time ordering $T$ is defined as

$$T(X_i(t)X_j(t')) = X_i(t)X_j(t') + \Theta(t' - t)X_j(t')X_i(t) = X_i(t)X_j(t') - \Theta(t' - t)[X_i(t), X_j(t')]$$

$$= X_i(t)X_j(t') - \Theta(t' - t)\left([A_i(t), A_j(t')] + [A_j(t), A_i(t')]\right).$$

On the other hand, the normal ordering :: moves by definition the creation part to the left of the annihilation part, so that

$$\langle 0 | : X_i(t)X_j(t') : |0 \rangle = \langle 0 | X_i(t)X_j(t') - [A_i(t), A_j(t')] |0 \rangle = 0.$$
\[ T(X_i(t)X_j(t')) - :X_i(t)X_j(t'):\]

\[= \Theta(t - t') [A_i(t), A_j^\dagger(t')] + \Theta(t' - t) [A_j(t'), A_i^\dagger(t)], \quad (9) \]

which is proportional to the identity operator \( \mathbf{1} \) by assumption (4). Now sandwich eq. (9) between the bra \( \langle 0| \) and the ket \( |0\rangle \). Since the rhs. of eq. (9) is proportional to the identity operator \( \mathbf{1} \), the unsandwiched rhs. must be equal to the sandwiched rhs. times the identity operator \( \mathbf{1} \). Hence also the unsandwiched lhs. of eq. (9) must also be equal to the sandwiched lhs. times the identity operator \( \mathbf{1} \). This yields eq. (5).

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1 The operators \( A_i \) and \( A_j^\dagger \) need not be Hermitian conjugates in what follows.

(i) Thanks a lot @Qmechanic, this clear proof is exactly what I needed. The only thing left to do for me now is to check that the preconditions are valid in my specific case. - Dilaton
@Qmechanic, for the operators \( A \) and \( B \), is the equation \( \langle 0|A + B|0\rangle = \langle 0|A|0\rangle + \langle 0|B|0\rangle \) satisfied? If it is satisfied, then we can sandwich the lhs. of eq. (1) and the equation simply becomes \( \langle 0| :X_i(t)X_j(t') :|0\rangle = 0 \). But such equation sometimes fails. - Wein Eld
I updated the answer. It seems you are talking about situations where one uses definitions of Fock vacuum and normal order that are not properly compatible/adjusted to each other. - Qmechanic