DMTH237 — Assignment 2 — Solutions

Linear systems, Reduction of Finite State Machines

Due 2:00 pm, Friday 5 April 2013

DMTH237 Assignment 2

Notes for markers: (30 marks) *Marks as shown for each question.*

1. For which values of the constant $k \in \mathbb{R}$ does the system $\begin{cases} x - y = 2 \\ 3\,x - 3\,y = k \end{cases}$ have no solution? Exactly one solution? Infinitely many solutions? Explain your reasoning.

   Solution: Row reducing the augmented matrix associated with the problem leads to

$$\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & -3 & k \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & k-6 \end{array}\right).$$

   The last column is a pivot column when $k \neq 6$, and so there exists no solution for these values of k. If on the other hand $k = 6$ then y is a free variable, and the solutions are given by $(x\,, y) = (2 + \lambda\,, \lambda)$, with $\lambda \in \mathbb{R}$ a parameter for the solution set. Hence there are infinitely many solutions in this case. Geometrically, the two lines corresponding with the two equations are parallel in $\mathbb{R}^2$, so there are only solutions if the lines are coincident, in which case there are infinitely many solutions.

   Notes for markers: (2 marks)

2. Solve the following (related) systems of linear equations by row reduction.

$$\text{(a)} \quad \begin{cases} x - 2\,y + z - 4\,w = 1 \\ x + 3\,y + 7\,z + 2\,w = 2 \\ x - 12\,y - 11\,z - 16\,w = 5 \end{cases} \qquad \text{(b)} \quad \begin{cases} x - 2\,y + z - 4\,w = 1 \\ x + 3\,y + 7\,z + 2\,w = 2 \\ x - 12\,y - 11\,z - 16\,w = -1 \end{cases}$$

   Make sure to verify that any 'solutions' you find are indeed solutions of the given system of equations.

   Solution: Row reduction of the augmented matrix associated with the first given system of three linear equations yields the sequence of similarities:

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right).$$

   Since the last column is pivotal, the given system has no solutions.
   Geometrically, the given hyperplanes in 4D have no point which is common to all three of them.

   Row reduction of the augmented matrix associated with the second given system of three linear equations yields the sequence of similarities:

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

   In this case, there are two free variables (the third and the fourth), and so we have an infinite two-parameter set of solutions. Choosing arbitrary values for these two variables, we can then solve for the second and the first variable, respectively. Doing so, we find all the solutions to be given by

$$\begin{cases} x = 1 - \alpha + 4\,\beta + \frac{2}{5}\,(1 - 6\,\alpha - 6\,\beta) = \frac{1}{5}\,(7 - 17\,\alpha + 8\,\beta) \\ y = \frac{1}{5}\,(1 - 6\,\alpha - 6\,\beta) \\ z = \alpha \\ t = \beta \end{cases}$$

   where α and β are arbitrary real numbers.

   Notes for markers: (5 marks)

3. Use the method of Gaussian elimination to find the inverse of each of the following matrices, provided the inverse exists.

   (a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ **Solution:** According to the method of Gaussian elimination, we have to apply elementary row operations to the given matrix; if we obtain the identity matrix for the reduced row echelon form, the given matrix is invertible, otherwise it is not. The same sequence of elementary row operations performed on the identity matrix of the appropriate dimension yields the inverse of the matrix, if it exists. A convenient way to do this is to append the identity matrix to the given matrix. We then obtain successively

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3/2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

   and so in this case we have: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}$.

   (b) $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ **Solution:** Only square matrices can have inverses, and so this matrix is *not invertible.*

   (c) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 5 \\ 0 & -3 & -8 \end{pmatrix}$ **Solution:** Here, an application of the process yields

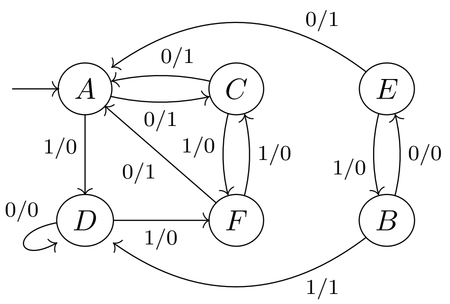
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 5 & 0 & 1 & 0 \\ 0 & -3 & -8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 8 & 1 & 1 & 0 \\ 0 & -3 & -8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 8 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

   and we can stop, since it will not be possible to further convert the given matrix into the identity matrix, the last row having only zeros on the left side. Therefore, the given matrix has no inverse.

   Notes for markers: (5 marks)

4. Suppose that $\mathcal{S} = \{A, B, \ldots, F\}$ is a set of states and $\mathcal{I} = \mathcal{O} = \{0, 1\}$ are the input and output alphabets for the Mealy machine described by the transition table below.

	Transition		Output	
	0	**1**	**0**	**1**
$\to A$	C	D	1	0
B	E	D	0	1
C	A	F	1	0
D	D	F	0	0
E	A	B	1	0
F	A	C	1	0



   (a) Find the output string corresponding to the input string '0110111011001', when starting in state A.

   Solution: The machine produces the string '1001000100110', finishing in state F.

   (b) Construct a state diagram corresponding to the machine.

   Solution: A possible layout for a state diagram is at right, above.

(c) Are there any non-accessible states? If so, remove them to get a (slightly) simpler reduced machine.

Solution: From the diagram it is clear that states B and E are inaccessible. Removing these from the state table leaves a 4-state machine, with no need to change anything in the 4 rows which remain. (The redundant lines are shown in grey, at left above.)

(d) With the (reduced) machine, consider the possible outputs for each input of a single '0' and a single '1'. Use the results to define the 0-equivalence classes. (At most there can be 4 classes. Why?) Number each state as 0, 1, 2 or 3 according to its 0-equivalence class.

Solution: With the reduced machine the output pairs are $A:(1,0)$, $C:(1,0)$, and $F:(1,0)$, so that states A, C and F have the same outputs, with $D:(0,0)$ being different. Call these three classes 0 and 1 respectively. At most there could be 4 different output pairs, but in fact only 2 occur here.

(e) Now, employing the 0-equivalence classes, use the procedure done in lectures to determine the 1-equivalence classes of states, then the 2-equivalence classes, etc. until the k-equivalence classes have stabilised. Identify any equivalent states. Remove redundant states, describing your result as a new transition table with the reduced number of states.

Solution: Using the 0-equivalence classes as the new transition states, then assigning the 1-equivalence classes, in the same way as for 0-equivalence, gives the extended table at left below — ignoring the bottom two rows corresponding to the inaccessible states B and D, and the parts shown in grey. One finds that the 1-equivalence classes are all distinct, so there are no equivalent states and there is no need to test further. A reduced machine is given at right below.

	Transition		Output			Transition			Transition		
	0	**1**	**0**	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	$\equiv_2$
$\rightarrow A$	C	D	1	0	0	0	1	0	1	2	0
C	A	F	1	0	0	0	0	1	0	1	1
D	D	F	0	0	1	1	0	2	2	1	2
F	A	C	1	0	0	0	0	1	0	1	1
B	E	D	0	1	2	0	1	3	4	2	3
E	A	B	1	0	0	0	2	4	0	4	4

Reduced machine:

	Transition		Output	
	0	**1**	**0**	**1**
$\rightarrow 0$	1	2	1	0
1	0	1	1	0
2	2	1	0	0

(f) Put the reduced Mealy machine into 'Standard Form'.

Solution: The reduced machine is already in standard form.

(g) Explore what happens if you do steps (d) and (e) without first having done step (c). Is it sufficient to do step (c) at the end, and still finish with the same transition table?

Solution: This is the purpose of the extra 2 rows in the equivalence table, at left above, and the parts in gray. The transitions for state B are different to those from every other state, so it has its own 0-equivalence class, but state E is in class 0. However, each has their own 1-equivalence class. The transitions to determine 2-equivalence are shown in grey italics. It really isn't necessary to have determined these unless the full machine was required.

Removing the inaccessible states *after* reduction gets back to the same reduced machine; it just takes a little bit more work.

(h) Convert the original Mealy machine into a Moore machine. Repeat steps (c), (d) and (e) to get a reduced machine with a minimum number of states. How does this machine compare with what you would have obtained by converting the final Mealy machine from step (e) into a Moore machine?

Solution:

	Transition			Transition			Transition			Transition		
	0	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	$\equiv_2$	**0**	**1**	$\equiv_3$
$\rightarrow A$	C'	D	0	1	0	0	2	1	0	4	2	0
B	E	D'	0	*inaccessible*								
C	A'	F	0	1	0	0	2	0	1	3	1	1
D	D	F	0	0	0	1	1	0	2	2	1	2
E	A'	B	0	*inaccessible*								
F	A'	C	0	1	0	0	2	0	1	3	1	1
A'	C'	D	1	1	0	2	2	1	3	4	2	3
B'	E	D'	1	*inaccessible*								
C'	A'	F	1	1	0	2	2	0	4	3	1	4
D'	D	F	1	*inaccessible*								
E'	A'	B	1	*inaccessible*								
F'	A'	C	1	*inaccessible*								

Reduced machine:

	0	**1**	
$\rightarrow 0$	4	2	0
1	3	1	0
2	2	1	0
3	4	2	1
4	3	1	1

Standard form:

	0	**1**		
$\rightarrow 0$	1	2	0	0
1	3	4	1	4
2	2	4	0	2
3	1	2	1	3
4	3	4	0	1

In the 12-state Moore machine, states B', E' and F' do not appear in the transition table, so are clearly not accessible. States B and E are inaccessible for the same reason as previously, which also renders D' to be inaccessible. With 2-equivalence all the states are in classes of their own, except for C and F, which remain equivalent under 3-equivalence. The reduced machine is given by ignoring the states marked as inaccessible, in the state table given above, and removing the 2nd instance of the equivalent state (namely state F). This is shown at top-right above.

Since the Moore machine never returns to state 0 (originally state A) it can be argued then that this can be removed, by shifting the initial state to 3 (state A', so also the initial state in the original Mealy machine). This reduces to a 4-state machine, with standard form as shown at right. However, this makes a difference to how the null-string is to be treated, if the Moore machine were regarded as an FSA. This technical point was not mentioned in lectures, when converting Mealy to Moore machines, so is not regarded as being strictly necessary here.

Standard form:

	0	**1**		
$\to$0	1	2	1	3
1	0	3	1	4
2	2	3	0	2
3	0	3	0	1

(i) Put the reduced Moore machine into 'Standard Form'.

Solution: In the reduced Moore machine, the states occur first in the order: $0, 4, 2, 3, 1$. By reordering the rows to match this, then relabelling state names, the machine converts to the standard form shown at lower-right above.

Notes for markers: (12 marks) *1 mark per part, except 2 marks for parts (e), (g), (h).*

5. (a) Which, if any, of the states in the following Finite State Acceptor are inaccessible?

	0	1	
$\to A$	C	F	
B	E	C	$*$
C	E	D	$*$
D	A	B	$*$
E	E	C	
F	C	A	

Solution: All the states in the given machine are accessible.

(b) What should one do with inaccessible states?

Solution: If there were any, they could be removed altogether.

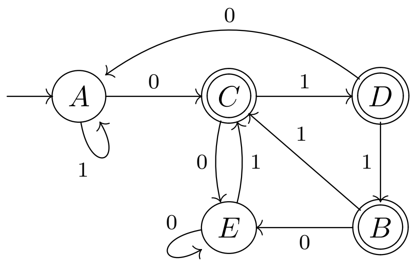
(c) Draw a state diagram for an FSA equivalent to this one, but with the smallest possible number of states.

Solution: States A and F are both non-accepting, and transition either the same or to each other. They can be confirmed to be indeed equivalent, with all other states in their own separate equivalence classes. A reduced machine, in which state F has been mapped to state A, with corresponding state diagram, is shown at left and middle below.

(d) Present the state table for the FSA in (c), in 'Standard Form'.

Solution: This is shown in the state table at right below.

	0	**1**	
$\to A$	C	A	
B	E	C	$*$
C	E	D	$*$
D	A	B	$*$
E	E	C	



	0	**1**	
$\to$0	1	0	
1	2	3	$*$
2	2	1	
3	0	4	$*$
4	2	1	$*$

Notes for markers: (6 marks) *1 mark per part, except 3 marks for part (c).*