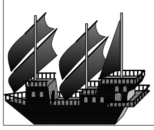
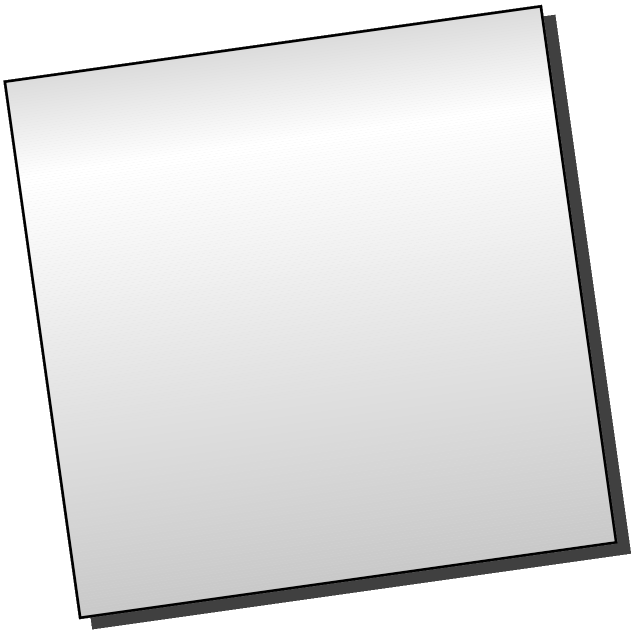




CHAPTER 1

TRANSVERSAL OPERATORS

1.1 Domain

This text is largely set in the space of *Lebesgue square-integrable functions* $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ (next).

Definition 1.1. *Let* $\mathbb{R}$ *be the set of real numbers,* $\mathscr{B}$ *the set of* BOREL SETS *on* $\mathbb{R}$*, and* μ *the standard* BOREL MEASURE *on . Let* $\mathbb{R}^{\mathbb{R}}$ *be the set of all functions with* DOMAIN $\mathbb{R}$ *and* RANGE $\mathbb{R}$.

DEF

The space of ***Lebesgue square-integrable functions*** $\boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)}$ *(or* $\boldsymbol{L}^2_{\mathbb{R}}$*) is defined as*

$$\boldsymbol{L}^2_{\mathbb{R}} \triangleq \boldsymbol{L}^2_{(\mathbb{R},\mathscr{B},\mu)} \triangleq \left\{ \mathrm{x} \in \mathbb{R}^{\mathbb{R}} \,\middle|\, \int_{\mathbb{R}} |\mathrm{x}|^2 \,\mathrm{d}\mu < \infty \right\}.$$

Furthermore, $\langle \cdot \mid \cdot \rangle$ *is the* INNER PRODUCT *induced by the operator* $\int_{\mathbb{R}} \mathrm{d}\mu$ *such that*

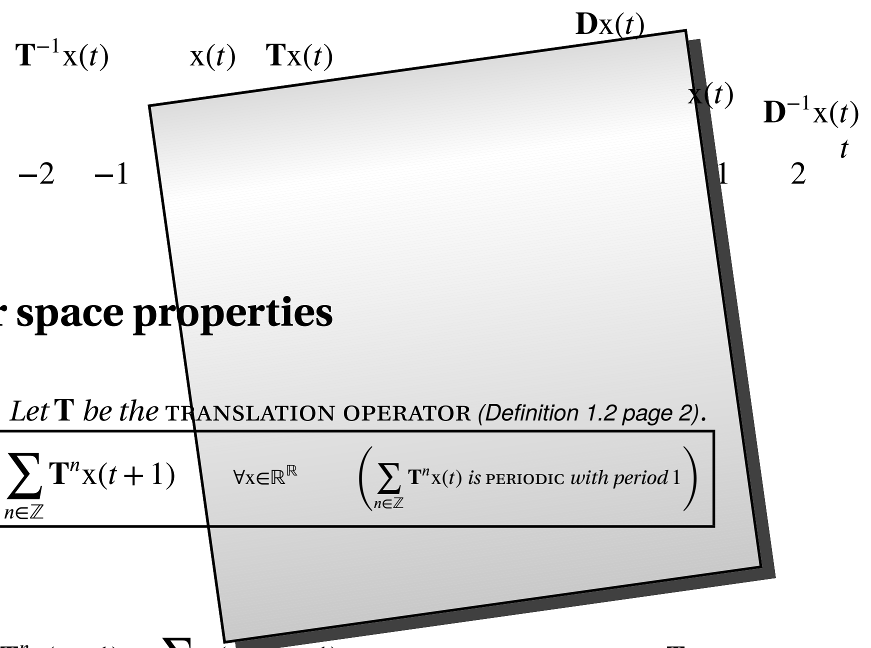
$$\langle \mathrm{x}(t) \mid \mathrm{y}(t) \rangle \triangleq \int_{\mathbb{R}} \mathrm{x}(t)\mathrm{y}^*(t)\,\mathrm{d}t.$$

1.2 Definitions

Much of the wavelet theory developed in this text is built on the **translation operator T** and the **dilation operator D** (next).

Definition 1.2. [1]

DEF	
1.	$\mathbf{T}$ *is the* ***translation operator*** *on* $\mathbb{R}^{\mathbb{R}}$ *if* $\quad \mathbf{T}\mathrm{x}(t) \triangleq \mathrm{x}(t-1) \quad \forall \mathrm{x}\in\mathbb{R}^{\mathbb{R}}$
2.	$\mathbf{D}$ *is the* ***dilation operator*** *on* $\mathbb{R}^{\mathbb{R}}$ *if* $\quad \mathbf{D}\mathrm{x}(t) \triangleq \sqrt{2}\mathrm{x}(2t) \quad \forall \mathrm{x}\in\mathbb{R}^{\mathbb{R}}$



1.3 Linear space properties

Proposition 1.1. *Let* $\mathbf{T}$ *be the* TRANSLATION OPERATOR *(Definition 1.2 page 2).*

PRP
$$\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathrm{x}(t) = \sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathrm{x}(t+1) \qquad \forall \mathrm{x}\in\mathbb{R}^{\mathbb{R}} \qquad \left(\sum_{n\in\mathbb{Z}} \mathbf{T}^n\mathrm{x}(t) \text{ is PERIODIC with period } 1\right)$$

✎PROOF:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \mathbf{T}^n \mathrm{x}(t+1) &= \sum_{n\in\mathbb{Z}} \mathrm{x}(t-n+1) && \text{by definition of } \mathbf{T} \text{ Definition 1.2 page 2} \\
&= \sum_{m\in\mathbb{Z}} \mathrm{x}(t-m) && \text{where } m \triangleq n-1 \\
&= \sum_{m\in\mathbb{Z}} \mathbf{T}^m \mathrm{x}(t) && \text{by definition of } \mathbf{T} \text{ Definition 1.2 page 2}
\end{aligned}$$

✎

Proposition 1.2. *Let* $\mathbf{D}$ *be the* DILATION OPERATOR *(Definition 1.2 page 2).*

PRP
$$\left\{\begin{array}{ll} 1. & \mathbf{D}\mathrm{x}(t) = \sqrt{2}\mathrm{x}(t) \quad \text{and} \\ 2. & \mathrm{x}(t) \text{ is CONTINUOUS} \end{array}\right\} \iff \{\mathrm{x}(t) \text{ is a CONSTANT}\} \qquad \forall \mathrm{x}\in\mathbb{R}^{\mathbb{R}}$$

[1] Walnut (2002): *An Introduction to Wavelet Analysis*, pages 79–80, ⟨Definition 3.39⟩
Christensen (2003): *An Introduction to Frames and Riesz Bases*, pages 41–42
Kammler (2008): *A First Course in Fourier Analysis*, page A-21
Bachman, Narici and Beckenstein (2000): *Fourier and Wavelet Analysis*, page 473
Packer (2004): *Applications of the Work of Stone and von Neumann to Wavelets*, page 260
Benedetto and Zayed (2004): *A Prelude to Sampling, Wavelets, and Tomography*, page
Heil (2011): *A Basis Theory Primer*, page 250, ⟨Notation 9.4⟩
Casazza and Lammers (1998): *Bracket Products for Weyl-Heisenberg Frames*, page 74
Goodman, Lee and Tang (1993a): *Transactions of the A.M.S. 338 [1993]*, page 639
Dai and Lu (1996): *Michigan Math. J. 43 [1996]*, page 81
Dai and Larson (1998): *Wandering vectors for unitary systems and orthogonal wavelets*, page 2

Proposition 1.3. *Let* **T** *be the* TRANSLATION OPERATOR *and* **D** *the* DILATION OPERATOR.

PRP
$$\begin{aligned}
\mathbf{T}\sum_{n\in\mathbb{Z}} \mathrm{x}(t) &= \sum_{n\in\mathbb{Z}} \mathbf{T}\mathrm{x}(t) \qquad \forall \mathrm{x}\in\mathbb{R}^{\mathbb{R}}\\
\mathbf{D}\sum_{n\in\mathbb{Z}} \mathrm{x}(t) &= \sum_{n\in\mathbb{Z}} \mathbf{D}\mathrm{x}(t) \qquad \forall \mathrm{x}\in\mathbb{R}^{\mathbb{R}}
\end{aligned}$$

✎PROOF:

$$\mathbf{T}\sum_{n\in\mathbb{Z}} \mathrm{x}(t) = \ldots$$ (Definition 1.2 page 2)

$$= \ldots$$ (Definition 1.2 page 2)

$$\mathbf{D}\sum_{n\in\mathbb{Z}} \mathrm{x}(t) = \sqrt{\ldots}$$ (Definition 1.2 page 2)

$$= \sum_{n\in\mathbb{Z}} \ldots$$

$$= \sum_{n\in\mathbb{Z}} \ldots$$ (Definition 1.2 page 2)

✎

In general the operators **T** and **D** are *noncommutative* ($\mathbf{TD} \neq \mathbf{DT}$), as demonstrated by Proposition 1.4 and by the following illustration.

1 x(t)
t
−2 −1 0 1 2

1.5
1.0 **TD**x(t)
0.5
t
−2 −1 0 1 2

1.5
1.0 **DT**x(t)
0.5
t
−2 −1 0 1 2

Proposition 1.4 (commutator relation). [2] *Let* **T** *be the translation operator and* **D** *be the dilation operator* (Definition 1.2 page 2).

PRP
$$\mathbf{T}^n\mathbf{D} = \mathbf{D}\mathbf{T}^{2n} \qquad \forall n \in \mathbb{Z}$$

✎PROOF:

[2] Christensen (2003): *An Introduction to Frames and Riesz Bases*, page 42, ⟨equation (2.9)⟩
Dai and Larson (1998): *Wandering vectors for unitary systems and orthogonal wavelets*, page 21
Goodman, Lee and Tang (1993a): *Transactions of the A.M.S. 338 [1993]*, page 641
Goodman, Lee and Tang (1993b): *Advances in Computational Mathematics 1*, page 110

1. Proof for $n = 0$:

$$\begin{aligned}
\mathbf{D}\mathbf{T}^{2n}\big|_{n=0} &= \mathbf{D}\mathbf{T}^{2\times 0} && \text{by } n=0 \text{ hypothesis}\\
&= \mathbf{D}\mathbf{I}\\
&= \mathbf{I}\mathbf{D}\\
&= \mathbf{T}^{0}\mathbf{D}\\
&= \mathbf{T}^{n}\mathbf{D}\big|_{n=0}
\end{aligned}$$

2. Proof for $n \neq 0$:

$$\begin{aligned}
\mathbf{D}\mathbf{T}^{2n}\mathrm{x}(t) &= \mathbf{D}\mathrm{x}(t-2n) && \text{by definition of } \mathbf{T} \text{ (Definition 1.2 page 2)}\\
&= \sqrt{2}\,\mathrm{x}(2t-2n) && \text{by definition of } \mathbf{D} \text{ (Definition 1.2 page 2)}\\
&= \sqrt{2}\,\mathrm{x}(2[t-n]) && \text{by \textit{distributivity} property of the field } (\mathbb{R},+,\times)\\
&= \mathbf{T}^{n}\sqrt{2}\,\mathrm{x}(2t) && \text{by definition of } \mathbf{T} \text{ (Definition 1.2 page 2)}\\
&= \mathbf{T}^{n}\mathbf{D}\mathrm{x}(t) && \text{by definition of } \mathbf{D} \text{ (Definition 1.2 page 2)}
\end{aligned}$$

✎

Proposition 1.5. *Let* $\mathbf{T}$ *be the translation operator and* $\mathbf{D}$ *the dilation operator (Definition 1.2 page 2).*

PRP					
	1.	$\mathbf{T}(\mathrm{xy})$	$=$	$(\mathbf{T}\mathrm{x})(\mathbf{T}\mathrm{y})$	$\forall \mathrm{x},\mathrm{y} \in \boldsymbol{L}^2_{\mathbb{R}}$
	2.	$\mathbf{D}(\mathrm{xy})$	$=$	$\frac{\sqrt{2}}{2}(\mathbf{D}\mathrm{x})(\mathbf{D}\mathrm{y})$	$\forall \mathrm{x},\mathrm{y} \in \boldsymbol{L}^2_{\mathbb{R}}$

✎PROOF:

$$\begin{aligned}
\mathbf{T}\big[\mathrm{x}(t)\mathrm{y}(t)\big] &= \mathrm{x}(t-1)\mathrm{y}(t-1) && \text{by definition of } \mathbf{T} \text{ (Definition 1.2 page 2)}\\
&= [\mathbf{T}\mathrm{x}(t)]\big[\mathbf{T}\mathrm{y}(t)\big] && \text{by definition of } \mathbf{T} \text{ (Definition 1.2 page 2)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}\big[\mathrm{x}(t)\mathrm{y}(t)\big] &= \sqrt{2}\mathrm{x}(2t)\mathrm{y}(2t) && \text{by definition of } \mathbf{D} \text{ Definition 1.2 page 2}\\
&= \frac{1}{\sqrt{2}}\Big[\sqrt{2}\mathrm{x}(2t)\Big]\Big[\sqrt{2}\mathrm{y}(2t)\Big]\\
&= \frac{\sqrt{2}}{2}[\mathbf{D}\mathrm{x}(t)]\big[\mathbf{D}\mathrm{y}(t)\big] && \text{by definition of } \mathbf{D} \text{ Definition 1.2 page 2}
\end{aligned}$$

✎

In a linear space, every operator has an *inverse*. Although the inverse always exists as a relation, it may not exist as a function or as an operator. But in some cases the inverse of an operator is itself an operator. The inverses of the operators $\mathbf{T}$ and $\mathbf{D}$ both exists as operators, as demonstrated by Proposition 1.6 (next).

1. By definition Definition 2.1, $\boldsymbol{L}^2_{\mathbb{R}}$ is *complete*.
2. In any metric space, (which includes all inner product spaces such as $\boldsymbol{L}^2_{\mathbb{R}}$), a *closed* subspace of a *complete* metric space is itself also *complete*.
3. In any *complete* metric space $\boldsymbol{X}$ (which includes all Hilbert spaces such as $\boldsymbol{L}^2_{\mathbb{R}}$), the two properties coincide—that is, a subspace is complete *if and only if* it is closed in the space $\boldsymbol{X}$.
4. So because $\boldsymbol{L}^2_{\mathbb{R}}$ is *complete* and each $\boldsymbol{V}_n$ is *closed*, then each $\boldsymbol{V}_n$ is also *complete*.

✏

2.1.4 Order properties

linearly ordered: A multiresolu
inclusion relation ⊆ forms the *l*
117) $\left(\left(\!\left(\boldsymbol{V}_n \right)\!\right), \subseteq\right)$, illustrated to th
tion A.9 page 119). Subspaces $\boldsymbol{V}_n$ in
That is, they contain more and m
and larger *n*—with the upper lim
ternatively, we can say that appro
yields greater "*resolution*" for incr

entire linear space

rger subspaces

aller subspaces

$\mathbf{0}$ smallest subspace

The *least upper bound* (*l.u.b.*) of the linearly ordered set $\left(\left(\!\left(\boldsymbol{V}_n \right)\!\right), \subseteq\right)$ is $\boldsymbol{L}^2_{\mathbb{R}}$:

$$\left(\bigcup_{n\in\mathbb{Z}} \boldsymbol{V}_n \right)^{-} = \boldsymbol{L}^2_{\mathbb{R}}.$$

The *greatest lower bound* (*g.l.b.*) of the linearly ordered set $\left(\left(\!\left(\boldsymbol{V}_n \right)\!\right), \subseteq\right)$ is $\mathbf{0}$:

$$\bigcap_{n\in\mathbb{Z}} \boldsymbol{V}_n = \mathbf{0}.$$

All linear subspaces contain the zero vector. So the intersection of any two subspaces must at least contain $\mathbb{0}$. If the intersection of any two linear subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ is exactly $\{\mathbb{0}\}$, then for any vector in the sum of those subspaces ($\boldsymbol{u} \in \boldsymbol{X} \dotplus \boldsymbol{Y}$) there are **unique** vectors $\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{y} \in \boldsymbol{Y}$ such that $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y}$. This is *not* necessarily true if the intersection contains more than just $\{\mathbb{0}\}$.

2.1.5 Bases for wavelet system

Definition 2.1 page 14 defines an MRA on the space $\boldsymbol{L}^2_{\mathbb{R}}$. The space $\boldsymbol{L}^2_{\mathbb{R}}$ is an example of a Hilbert space. A Hilbert space is a linear space equipped with an inner product and that is complete with respect to the topology induced by the inner product.

Example 2.1. In the *Haar* MRA, the scaling function $\phi(t)$ is the *pulse function*

$$\phi(t) = \begin{cases} 1 & \text{for} \quad 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

1

The scaling subspace V_0 is the span $V_0 \triangleq \operatorname{span}\{\mathbf{T}^n\phi\,|_{n\in\mathbb{Z}}\}$.

1 t

In the subspace V_n ($n \in \mathbb{Z}$) the scaling functi… …2)^n

that

$$\mathbf{D}^n\phi(t) = \begin{cases} \left(\sqrt{2}\right)^N & \ldots \\ 0 & \ldots \end{cases}$$

$\left(\frac{1}{2}\right)^n$ t

The scaling subspace V_n is the …

Note that $\|\mathbf{D}^n\phi\|$ for each resol…

$$\|\mathbf{D}^n\mathbf{T}^m\phi\|^2 = \ldots 1 \text{ page } 8$$

$$= \int \ldots$$

$$= 1$$

1

Let $\mathrm{f}(t) = \sin(\pi t)$. Suppose we want to approximate $\mathrm{f}(t)$ in the subspaces $V_0, V_1, V_2, \ldots$.

−3 −2 −1 1 2 3

−1

$\frac{2}{\pi}$

The values of the transform coefficients for the subspace V_n are given by

n

−3 −2 −1 $\frac{-2}{\pi}$ 1 2 3

$$\begin{aligned}
\left[\mathbf{R}_n\mathrm{f}(t)\right](m) &= \frac{1}{\|\mathbf{D}^n\mathbf{T}^m\phi\|^2}\left\langle \mathrm{f}(t) \,|\, \mathbf{D}^n\mathbf{T}^m\phi\right\rangle \\
&= \frac{1}{\cancelto{1}{\|\mathbf{D}^n\mathbf{T}^m\phi\|^2}}\left\langle \mathrm{f}(t) \,|\, \mathbf{D}^n\phi(t-m)\right\rangle && \text{by definition of } \mathbf{T} \text{ (Definition 1.2 page 2)} \\
&= \left\langle \mathrm{f}(t) \,|\, \left(\sqrt{2}\right)^N \phi\left(2^N t - m\right)\right\rangle && \text{by definition of } \mathbf{D} \text{ (Definition 1.2 page 2)} \\
&= \left(\sqrt{2}\right)^N \left\langle \mathrm{f}(t) \,|\, \phi\left(2^N t - m\right)\right\rangle \\
&= \left(\sqrt{2}\right)^N \int_{\left(\frac{1}{2}\right)^N m}^{\left(\frac{1}{2}\right)^N (m+1)} \mathrm{f}(t)\,\mathrm{d}t
\end{aligned}$$

$$= \left(\sqrt{2}\right)^N \int_{\frac{n}{2^N}}^{\frac{n+1}{2^N}} \sin(\pi t)\; \mathrm{d}t$$

$$= \left(\sqrt{2}\right)^N \left(-\frac{1}{\pi}\right) \cos(\pi t)\Big|_{\left(\frac{1}{2}\right)^N m}^{\left(\frac{1}{2}\right)^N (m+1)}$$

$$= \frac{\left(\sqrt{2}\right)^N}{\pi} \left[\cos\left(\pi \frac{n}{2^N}\right) - \cos\left(\ldots\right.\right.$$

And the approximation $\mathbf{A}_n$ … lternatively, the *projection* of $\mathrm{f}(t)$ *onto* the space $\boldsymbol{V}_n$ is)

$$\mathbf{A}_n \mathrm{f}(t) = \sum_{m\in\mathbb{Z}} \langle \ldots$$

$$= \frac{\left(\sqrt{2}\right. \ldots}{\pi} \ldots {}^N \phi\left(2^N t - m\right)$$

$$= \frac{2^N}{\pi} \sum_{m\in\mathbb{Z}} \ldots$$

The transforms into the subspa…, $\boldsymbol{V}_1$, and $\boldsymbol{V}_2$, as well as the approximations in those subspaces are as illustrated next:

subspace	transform	approximation
$\boldsymbol{V}_0$	$\frac{2}{\pi}$ -3 -2 -1 $\frac{-2}{\pi}$ 1 2 3 n	$\frac{2}{\pi}$ -3 -2 -1 $\frac{-2}{\pi}$ 1 2 3 t
$\boldsymbol{V}_1$	$\frac{\sqrt{2}}{\pi}$ n $\frac{-\sqrt{2}}{\pi}$	$\frac{\sqrt{2}}{\pi}$ t $\frac{-\sqrt{2}}{\pi}$
$\boldsymbol{V}_2$	n	t