chapterIntroduction

The stability of nonlinear waves has a distinguished history and an abundance of richly structured yet accessible examples, which makes it not only an important subject but also an ideal training ground for the study of linear and nonlinear ~~PDEs~~partial differential equations (PDEs). While the ~~“modern”~~ ??modern?? approach to the stability of nonlinear waves can be traced back to the key papers of Joseph Boussinesq in the 1870s, the field has experienced tremendous growth ~~in~~ over the past 30 years. Some of the growth was stimulated by T. B. Benjamin's 1972 ~~publication “~~paper, ??The stability of solitary waves~~” whose~~ ,?? which presented a treatment of the Korteweq–de Vries equation that in many ways placed meat on the bones of Boussinesq's ideas. A more recent avenue of growth stems from the development of dynamical systems ideas, which provide a rich ~~compliment~~ complement to the functional analytic approach. In many ways these developments were stimulated by the pioneering work of Alexander, Jones, and Gardner ~~who recast the Evans function in a dynamical~~ systems language. The subsequent synergy between dynamical systems and functional analysis has yielded a burst of activity and produced a unified framework for the study of stability and bifurcation in nonlinear waves.

Many graduate students in applied mathematics have been exposed to the key ideas of dynamical systems and functional analysis by the end of their first year; however, these fields are typically presented as unrelated. Within the context of nonlinear waves, the goal of this book is to show how the tools of dynamical systems provide a rich illumination of the abstract ideas of functional analysis. However, the simultaneous application of these two fields requires some sophistication. Our approach is to motivate the abstract framework by first working through detailed examples ~~which~~ that serve to illustrate the moving parts, with the broader framework subsequently presented as a generalization of a familiar process. As much as possible, we have emphasized the structure of the framework: showing that stability and bifurcation can be understood through the dynamical systems which characterize the underlying equilibria. We have avoided the use of exact solutions of ~~ODEs~~ordinary differential equations (ODEs), whose detailed calculus tends to obfuscate the structure behind the ideas.

Towards this goal, autorefch:background provides a summary of background material; essentially, ~~that which~~ what is assumed of a mathematics graduate student who has completed a year of graduate PDE and functional analysis. We understand that this knowledge may be incomplete and therefore we have made an effort to integrate its application into the later chapters in a self-contained manner. In many ways autorefch:spectra is the beginning of the book, applying dynamical systems ideas to illuminate a fundamental result of functional analysis: the Fredholm classification of linear differential operators on the line. The Fredholm alternative is the underpinning of many existence and bifurcation results, and its proof through dynamical systems techniques is instructive. This approach places the solvability question for a "linear system of differential equations subject to boundary conditions" inside of the bigger box of "all flows generated by the equivalent dynamical system~~".~~." The task is to classify all solutions of the dynamical system, and then ask which satisfy the boundary conditions. This recalls the classical construction of a Green's function on a bounded domain, but the extension of the ideas to the unbounded domain, particularly the classification step, ~~lead~~ leads naturally to the idea of the Evans function, and provides a concrete formulation of the essential spectrum of important classes of linear differential operators.

After autorefch:spectra, the flow of the book branches, leading either to a study of nonlinear stability and bifurcation via functional analytic techniques in Chaps. **??**–**??**, or directly to the Evans function and the key applications of eigenvalue perturbation within the essential spectrum in Chaps. **??**–**??**. The functional analytic tract is inspired qualitatively by the work of Benjamin, and more quantitatively by the work of Grillakis ~~, Shatah, and Strauss, who provided the first comprehensive treatment of the~~ et al.stability of Hamiltonian systems in the presence of symmetry. Our approach introduces the unifying idea of a constrained operator and a quantitative measure of the impact of linear constraint upon eigenvalue count. The constrained operator approach naturally abuts the Hamiltonian index theory, and plays a central role in the stability of critical points of Hamiltonian systems. Chapter **??** presents an overview the properties of $calC^0$ and analytic semigroups, with attention to the role of symmetries in generating a point spectrum, and the idea that spectral stability in many situations implies orbital stability of a manifold of equilibria. In autorefch:hamsys we adapt these ideas to the Hamiltonian framework, using an adaptation of Benjamin's proof of the orbital stability of the solitary solution of the KdV equation to illustrate the idea of a constrained operator within a concrete framework. The framework is substantially generalized in autorefs:hamsys, and is carried forward to Chaps. **??** and **??** with slight modification. In autorefch:ls we present the formal perturbation structure of regular spectra of linear operators, with concrete examples ~~which~~ that illustrate the major stability results of Chaps. **??** and **??**. Chapter **??** gives a detailed analysis of index theory for Hamiltonian systems, introducing the Hamiltonian–Krein index, which enumerates the potentially destabilizing point spectrum, and the Krein signature. In particular, we develop an instability criterion ~~which~~ that is complementary to the nonlinear stability results of autorefch:hamsys. We also discuss applications to bifurcation of point spectra with ~~non-trivial~~ nontrivial Jordan block structure under both ~~symmetry breaking~~ symmetry-breaking Hamiltonian perturbations and symmetry ~~preserving~~ -preserving non-Hamiltonian perturbations.

The second part of the book develops the Evans function, an analytic function of the spectral parameter of an associated eigenvalue problem, ~~whose zeros~~ with zeros that coincide with the point spectrum of the operator when they are within the natural domain of the Evans function. The Evans function has analytic extensions beyond its natural domain, generically some distance into the essential spectrum. This extension plays a fundamental role in understanding bifurcations associated ~~to~~ with point spectra which are ejected from the essential spectrum of the linearized operator under perturbation.

The machinery required to develop the Evans function is ~~non-trivial~~nontrivial; we first consider the simpler context of boundary-value problems on a finite domain in autorefch:evansbv. In this context the Evans function resembles a classical Wronskian, being an entire function of the spectral parameter. The issues of branch points, branch cuts, and Riemann surfaces are deferred until autorefch:evanssl which addresses second-order Sturm–Liouville operators on the line. This requires a re-examination of the issue addressed in autorefch:spectra: the classification of all the solutions of a given dynamical system by their asymptotic behavior, which evokes the Jost functions of quantum mechanics. Even for this restricted class of problem one finds the full richness of the eigenvalue problems: branch points and branch cuts of the Evans function, and its extension beyond the natural domain and onto Riemann sheets. Within the context of second-order differential operators we address two important problems: the detection of real eigenvalues associated with instabilities, and the tracking of eigenvalues as they enter and leave the branch cuts. We also discuss the connection of the absolute spectrum to the large domain limit of the bounded domain problem for both separated and periodic boundary conditions. Chapter **??** addresses issues arising in the extension of the Evans function to higher-order linear operators, and incorporates several substantial examples that display the breadth of possible applications.

There are many relevant topics ~~which~~ that are not touched upon in this book. All issues associated ~~to~~ with multiple scales and (geometric) singular perturbation theory have been set aside. Eigenvalue problems can also be considered through a topological version of the Evans function, e.g., see ~~,~~citepkapitula:eas96,bose:sot95,rubin:sba99,gardner:two89, or by decomposing the Evans function into the product of meromorphic functions, each of which is associated with a distinct time scale, e.g., see citepdoelman:asi02,doelman:sao98. The Green's function approach to nonlinear stability, pioneered by Howard and Zumbrun, is a natural ~~out-growth~~ outgrowth of the dynamical systems approach. Another natural extension is to weak and semi-strong interaction regimes of ~~multi-pulse~~ multipulse solutions e.g., see citepsandstede:eas97,sandstede:som98,sandstede
The most obvious limitation is that we have restricted ourselves to systems in one space dimension; however, these and other extensions are the domain of current research and do not seem ready for a unified treatment. The literature in the field of nonlinear waves is vast, and we have made an attempt to reference relevant papers at the end of each section; however, our effort is of necessity incomplete and we apologize in advance to the many people whose work we have not fully cited.

The content of the book has been used in a course for second-year graduate students at Michigan State University. It could also naturally be used to form two possible one-semester courses. The first course might emphasize the functional analytic approach, and include Chaps. **??**–**??**. The second could address the Evans function and dynamical systems approach, using Chaps. **??** and **??**–**??**. Both routes would be supplemented with material from Chap. **??** as needed, and of course from the instructor's own personal repertoire of examples.