Abstract We compute all countably infinitely many complex roots z of a pseudo-trionmial $z^{\alpha}+w\cdot z+1$ for arbitrary complex numbers α and w . We combine the use of Mishkovs formula and Egorychevs generalization of the Lagrange Inversion Formula with formulae for all $|n|$ roots of the polynomial $z^n+w\cdot z+1$ for a given integer n . We point to ongoing efforts to generalize Mishkovs formula in order to compute all the roots of pseudopolynomials $\sum\limits_{i=1}^{n} w_i\cdot z^{\alpha_i}$ or any expression involving a finite number of parameters $\{\alpha_i\}_{i=1}^{n}$.

Introduction Given a complex-valued function $w=g(z)$ of a complex variable z with $g(0)=0$, which is analytic around 0, the Lagrange Inversion Formula gives an expression for a Maclaurin power series $z=f(w)$ with $f(0)=0$ which is the inverse of a g . However, Lagrange does not give us the other (in general) infinitely many inverses $\{f_k(w)\}_{k\in\mathbb{N}}$ of g .

Let α be a constant indeterminate over a number field of characteristic zero. For any complex number z the symbol z^{α} means the germ of $\exp(\alpha\cdot\ln(z))=\sum\limits_{n\in\mathbb{N}}\frac{1}{n!}\alpha^n(\ln(z))^n$. Define $G(t,\alpha)\equiv t^{\alpha}+t+1$. Suppose $G(z,\alpha)=0$.

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If G and t are scalars, $(x_1(t),...,x_r(t))$ is an r -vector and $G(x_1(t),...,x_r(t))$ is a composite function, for which all necessary derivatives are defined, D is the differential operator with respect to t , then $D^nG(x_1(t),...,x_r(t))=\sum\limits_{0}\sum\limits_{1}\cdots\sum\limits_{n}\frac{n!}{\left(\prod\limits_{i=1}^{n}(i!)^{k_i}\right)\cdot\left(\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{r}q_{i,j}!\right)}\cdot\frac{\partial^k G}{\prod\limits_{j=1}^{r}\partial x_j^{p_j}}\cdot\Upsilon_q$ where $\Upsilon_q=\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{r}\left(\frac{d^i x_j}{dt^i}\right)^{q_{i,j}}$ where the respective sum $\sum\limits_{i}$ for a given $i\in[n]$ is over all nonnegative integer solutions of the Diophantine equation $k_i=\sum\limits_{j=1}^{r}q_{i,j}$ and $\sum\limits_{0}$ is over all nonnegative integer solutions of $\sum\limits_{i=1}^{n}i\cdot k_i=n$. The order p_j of the partial derivative of G with respect to x_j satisfies $p_j=\sum\limits_{i=1}^{n}q_{i,j}$ (Mishkov (2.3) p483). The total order k of the derivative of G with respect to all x_j s satisfies $k=\sum\limits_{j=1}^{r}p_j=\sum\limits_{i=1}^{n}k_i$. In ongoing research we need to extend the domain slightly of q to include. Mishkovs Theorem can be expressed without specifying the k_i s nor the p_j s. In other words,

Corollary 1. Mishkovs Theorem may be rewritten more succinctly as $D^nG(x_1(t),...,x_r(t))=\sum\limits_{q}\frac{n!}{\left(\prod\limits_{i=1}^{n}(i!)^{k_i}\right)\cdot\left(\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{r}q_{i,j}!\right)}$ where the sum is over all maps $q:[n]\times[r]\to[n]_0$ which satisfy $\sum\limits_{i=1}^{n}i\cdot\sum\limits_{j=1}^{r}q_{i,j}=n$.

Corollary 2. Let $r=2$, $x_1=\alpha$, $x_2=z$ in Mishkovs theorem. Then $D^nG(\alpha(t),z(t))=$

$\sum\limits_{q}\frac{n!}{\left(\prod\limits_{i=1}^{n}(i!)^{k_i}\right)\cdot\left(\prod\limits_{i=1}^{n}(q_{i,1}!q_{i,2}!)\right)}\cdot\frac{\partial^{\sum\limits_{i=1}^{n}q(i,1)+\sum\limits_{i=1}^{n}q(i,2)}G}{\partial\alpha^{\sum\limits_{i=1}^{n}q(i,1)}\partial z^{\sum\limits_{i=1}^{n}q(i,2)}}\cdot\prod\limits_{i=1}^{n}\left(\left(\frac{d^i\alpha}{dt^i}\right)^{q(i,1)}\left(\frac{d^i z}{dt^i}\right)^{q(i,2)}\right)$ where the sum is over all maps $q:[n]\times[2]\to[n]_0$ which satisfy $\sum\limits_{i=1}^{n}i\cdot(q_{i,1}+q_{i,2})=n$.

Commentary: the other Italian authors

Lemma 3. $\frac{d^n}{d\alpha^n}G(\alpha,z(\alpha))=$

$\sum\limits_{q}\frac{n!}{\left(\prod\limits_{i=2}^{n}(i!)^{q(i)}\right)\cdot\left((n-\sum\limits_{i=1}^{n}i\cdot q_i)!\right)\left(\prod\limits_{i=1}^{n}(q_i!)\right)}\cdot\frac{\partial^{n+\sum\limits_{i=1}^{n}(1-i)\cdot q(i)}G}{\partial\alpha^{n-\sum\limits_{i=1}^{n}i\cdot q(i)}\partial z^{\sum\limits_{i=1}^{n}q(i)}}\cdot\prod\limits_{i=1}^{n}\left(\left(\frac{d^i z}{d\alpha^i}\right)^{q(i)}\right)$ where we sum over all maps $q:[n]\to[n]_0$. Proof. Specialize $t\to\alpha$ in Corollary 2. Then $\frac{d^i\alpha}{dt^i}\to\frac{d^i\alpha}{d\alpha^i}$ and $\frac{d^i z}{dt^i}\to\frac{d^i z}{d\alpha^i}$. So $\frac{d^n}{d\alpha^n}G(\alpha,z(\alpha))=$

$\sum\limits_{q}\frac{n!}{\left(\prod\limits_{i=1}^{n}(i!)^{k_i}\right)\cdot\left(\prod\limits_{i=1}^{n}(q_{i,1}!q_{i,2}!)\right)}\cdot\frac{\partial^{\sum\limits_{i=1}^{n}q(i,1)+\sum\limits_{i=1}^{n}q(i,2)}G}{\partial\alpha^{\sum\limits_{i=1}^{n}q(i,1)}\partial z^{\sum\limits_{i=1}^{n}q(i,2)}}\cdot\prod\limits_{i=1}^{n}\left(\left(\frac{d^i\alpha}{d\alpha^i}\right)^{q(i,1)}\left(\frac{d^i z}{d\alpha^i}\right)^{q(i,2)}\right)$

Since $i>1\Rightarrow\frac{d^i\alpha}{d\alpha^i}=0$ and since $q(i,1)>0\Rightarrow 0^{q(i,1)}=0$, we may ignore all maps q such that

$i > 1 \Rightarrow q(i,1) > 0$. So the condition $\sum_{i=1}^{n} i \cdot (q_{i,1} + q_{i,2}) = n$ is reduced to $q_{1,1} + \sum_{i=1}^{n} i \cdot q_{i,2} = n$. Since $i = 1 \Rightarrow \frac{d^i \alpha}{d\alpha^i} = 1$ and $1^{q(1,1)} = 1$ for any value of $q(1,1)$ we may rewrite q as a map with just one argument, so $q : [n] \to [n]_0$. Since $q(1,1)$ can range over all $[n]_0$, our condition $\sum_{i=1}^{n} i \cdot q_{i,2} = n - q_{1,1}$ is further simplified to $\sum_{i=1}^{n} i \cdot q_i$ ranging over all $[n]_0$. So $\frac{d^n}{d\alpha^n} G(\alpha, z(\alpha)) =$

$\sum_q \frac{n!}{\left(\prod_{i=2}^{n} (i!)^{q(i)}\right) \cdot \left((n - \sum_{i=1}^{n} i \cdot q_i)!\right) \left(\prod_{i=1}^{n} (q_i!)\right)} \cdot \frac{\partial^{n + \sum_{i=1}^{n} (1-i) \cdot q(i)} G}{\partial \alpha^{n - \sum_{i=1}^{n} i \cdot q(i)} \partial z^{\sum_{i=1}^{n} q(i)}} \cdot \prod_{i=1}^{n} \left(\left(\frac{d^i z}{d\alpha^i} \right)^{q(i)} \right)$ where we sum over all maps $q : [n] \to [n]_0$. Q.E.D.

Lemma 4. If $G(\alpha, z) = 0$, then for each $n \in \mathbb{N}$ we have $0 = \sum_q \frac{n!}{\left(\prod_{i=2}^{n} (i!)^{q(i)}\right) \cdot \left((n - \sum_{i=1}^{n} i \cdot q_i)!\right) \left(\prod_{i=1}^{n} (q_i!)\right)} \cdot \frac{\partial^{n + \sum_{i=1}^{n} (1-i) \cdot q(i)} G}{\partial \alpha^{n - \sum_{i=1}^{n} i \cdot q(i)} \partial z^{\sum_{i=1}^{n} q(i)}}$ where we sum over all maps $q : [n] \to [n]_0$.

Proof. All total derivatives of $G(\alpha, z) = 0$ with respect to α are zero. The result follows trivially from Lemma 3. Q.E.D.

Definition 5. Define the multivariable polynomial $H_n(\alpha, u_1, ..., u_n) \equiv . \frac{1}{G_z} \sum_q \frac{n!}{\left(\prod_{i=2}^{n} (i!)^{q(i)}\right) \cdot \left((n - \sum_{i=1}^{n} i \cdot q_i)!\right) \left(\prod_{i=1}^{n} (q_i!)\right)} \cdot \frac{\partial^n}{\partial \alpha^{n-}}$ where we leave the partial derivative of G with respect to α and z as an indeterminate function of α and z . So, strictly, Definition 5 is not polynomial in α . Since $G(\alpha, z) = 0 \Rightarrow \forall n \in \mathbb{N} : \frac{1}{G_z} \frac{d^n}{d\alpha^n} G(\alpha, z) = 0 \Rightarrow \forall n \in \mathbb{N} : H_n(\alpha, \frac{dz}{d\alpha}, ..., \frac{d^n z}{d\alpha^n}) = 0$, in other words $\forall n \in \mathbb{N} : H_n = 0$ after specialize $\forall i \in [n] : u_i \to \frac{d^i z}{d\alpha^i}$. Definition 6. A singularity of $G(z, \alpha)$ is a value of α such that at least one of the roots z of $G(z, \alpha) = 0$ is infinity, or two or more of the roots z of $G(z, \alpha) = 0$ are equal. In other words, one of the roots z of $G(z, \alpha) = 0$ is also a root of $G_z \equiv \frac{\partial G(z, \alpha)}{\partial z} = 0$.

Theorem 7. Egorychev Lagrange Inversion

$\frac{\partial^{m+n}}{\partial \alpha^m \partial u^n} \left[u_j \cdot \prod_{k=1}^{n} \left(H_k(\alpha, u) - u_k \right)^{\beta(k)} \right]$

Proof. The highest order of the derivative of z with respect to α in $\frac{d^n}{d\alpha^n} G(\alpha, z(\alpha))$ is n , and the highest power of $\frac{d^n z}{d\alpha^n}$ in the summation in Lemma 4 is 1, which occurs when $i < n \Rightarrow q(i) = 0$ and $q(n) = 1$ and $\sum_{i=1}^{n} i \cdot q_i = n \cdot q_n = n \cdot 1 = n$, the coefficient of $\frac{d^n z}{d\alpha^n}$ in $\frac{d^n}{d\alpha^n} G(\alpha, z(\alpha))$ is $\frac{n!}{(n!) 1 \cdot 1!} \cdot \frac{\partial G}{\partial z} = G_z$. So, the coefficient of $\frac{d^n z}{d\alpha^n}$ in H_n is 1. First check that for each $n \in \mathbb{N}$ and each $k \in [n]$ that $H_k(\alpha, u_1, ..., u_n)$ satisfies $\frac{\partial}{\partial u_j} H_k(\alpha, u_1, ..., u_n) \Big|_{u(1) = \cdots = u(n) = 0} = \delta_{j,k}$.

So $G(\alpha, z) = 0 \Rightarrow \forall n \in \mathbb{N} : \frac{1}{G_z} \frac{d^n}{d\alpha^n} G(\alpha, z) = 0 \Rightarrow \forall n \in \mathbb{N} : H_n(\alpha, z) = 0$. We must compute

$$\frac{\partial^{m + \sum_i \beta(i)}}{\partial \alpha^m \prod_{i=1}^{n} \partial u_i^{\beta(i)}}$$

$$\left[u_n \cdot \prod_{k=1}^{n} \left(\frac{1}{G_z} \sum_q \frac{k!}{\left(\prod_{i=1}^{k} (i!)^{q(i)}\right) \cdot \left((k - \sum_{i=1}^{k} i \cdot q_i)!\right) \left(\prod_{i=1}^{k} (q_i!)\right)} \cdot \frac{\partial^{k + \sum_{i=1}^{k} (1-i) \cdot q(i)} G}{\partial \alpha^{k - \sum_{i=1}^{k} i \cdot q(i)} \partial z^{\sum_{i=1}^{k} q(i)}} \cdot \prod_{i=1}^{k} \left(u_i^{q(i)} \right) - u_k \right)^{\beta(k)} \right]$$

and then evaluate this partial derivative at $u_1 = \cdots = 0 = \cdots = u_n$. Commentary

Theorem 5. Taylor series

Any root z of a given function $G(z, \alpha)$ of z and a single complex-valued parameter α is given by

$z = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n z}{d\alpha^n}\Big|_{\alpha=m} \cdot (\alpha - m)^n$ whose radius of convergence is the distance from α to the nearest singularity of $G(z, \alpha)$.

Corollary. For $G(z, \alpha, w) \equiv z^\alpha + w \cdot z - 1$,

Commentary

Theorem 5. Belidarnelli formula for resolvent of a trinomial

Mishkov, Rumen L. Generalization of the Formula of Faa Di Bruno for a Composite Function with a Vector Argument Internat. J. Math. & Math. Sci. Vol 24, No. 7 (2000) 481-491.