Peter's Problem

A company representative's weekly work schedule requires a visit to customers in each of the following 21 towns:

1. Athlone, Co. Westmeath
2. Balbriggan, Co. Dublin
3. Carrick-on-Shannon, Co. Leitrim
4. Drogheda, Co. Louth
5. Enniscorthy, Co. Wexford
6. Freshford, Co. Kilkenny
7. Greystones, Co. Wicklow
8. Headford, Co. Galway
9. Kilcullen, Co. Kildare
10. Lismore, Co. Waterford
11. Mountrath, Co. Laois
12. Nenagh, Co. Tipperary
13. Oranmore, Co. Galway
14. Portlaoise, Co. Laois
15. Rathkeale, Co. Limerick
16. Swinford, Co. Mayo
17. Tullamore, Co. Offaly
18. Urlingford, Co. Kilkenny
19. Virginia, Co. Cavan
20. Westport, Co. Mayo
21. Youghal, Co. Cork

The attached table of distances gives the Automobile Association preferred routes and distances between each of the towns listed.

NOTE: Identify and list the reference for all details used in your solution.

Questions

1. Using the **table of distances**, plan and write down the shortest route that should be taken to minimise the distance travelled to visit all of the towns in one complete circuit, starting in Athlone as home-base and returning to that base following the final visit.

2. Using a suitable chart, table or otherwise, illustrate the route you have chosen, itemising the distance between each town in turn.

3. Using the Statistics section of the SIMI (Society of the Irish Motor Industry) website, find the most popular 152-registered car make and model - sold in the Republic of Ireland.

4. Using the combined driving performance claimed by the manufacturer in their Product Guide for the car model, find the amount of petrol, in litres, used by the car on those trips (the rep is provided with a 1.4 litre, 150HP, manual, 5-door car of the make and model identified in Q3).

5. If the rep was driving the 1.6 litre manual, 90HP, 5 door diesel version of the **same** car model, calculate the amount of diesel, in litres, used by this car, again using the combined driving performance claimed by the manufacturer.

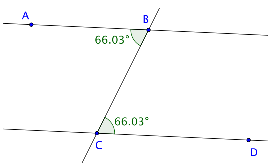
6. If diesel is 10.2 cent per litre cheaper than petrol, calculate the difference between the cost of fuelling the two cars for all 21 journeys, rounded to the nearest 5 cent.

7. Again, using the manufacturers statistics, calculate which car is cleaner for the environment in terms of CO2 emissions by using the CO2 emissions figures provided in the Product Guide for each vehicle and state the difference between these emissions for the total journey, correct to the nearest gram.

Experiment, Surprise and the need for Proof

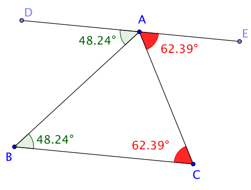
Dynamic geometry software cannot be used as a substitute for proof. It can be used, when guided by the teacher, to search for things that might very well be worth proving. It is not possible to recreate five thousand years of mathematics in four or five 35 minute sessions a week. It is possible to to excite students and get them to experiment.

The converse of Theorem 3 says that *If two lines are parallel, then any transversal will make equal alternate angles with them.* To many children that sentence is meaningless. They may understand the words but not what is being said.



Here is where dynamic geometry can be used very successfully. This applet is designed so that the student can play with the lines. The lines can be moved and their direction can be changed. What is important here is the ability to move the lines. As a child slides the line A down or the line D up it will become obvious that, in some sense, the two lines are the same. Then the equality of the angles jumps out and it makes sense to try to prove it.

Theorem 4 says *The angles in any triangle add to 180°*. To most children this makes no real sense yet. The proof is difficult for them because if lines are extended the triangle is disguised whereas if they are not then children find it difficult to associate the side of a triangle as part of an infinite line. The applet here allows the teacher show and hide lines. The student can change the shape of the triangle.



Once you have a number of theorems proved then the power of proof is easier to show. For example Theorem 13, which says *If two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, then their sides are proportional*, needs Theorem 12 in its proof. Theorem 12 needs Theorem 11. Theorem 11 needs Theorem 9. Theorem 9 needs Theorem 3. Theorem 3 is based on axioms and definitions. So the chain of deduction is built up.
Surprise is a very useful tool. We are all very familiar with the results we want to introduce. Our students are not. Consider how to introduce children to the formula for the length of the circumference of a circle, $C = 2\pi r$. It is very easy to forget how surprising this result is. If you are dealing with First Year students try the following.

Requirements:

- Compass and Ruler
- Paper
- Thread

Instructions:

1. Use the ruler to fix the radius of a circle. You can draw any size circle you like provided it fits on the paper.
2. Use the thread to measure the circumference of the circle as well as you can.
3. Divide this answer by two.
4. Divide this new number by the radius of your circle. You will all get about the same answer and it will be a little bit bigger than three!

Your students will find this incredible. You are likely to get protests that “That’s impossible. All the circles are different.” The surprise is only starting, though. Wait about a week and then come back to investigate the area of a circle.

Requirements:

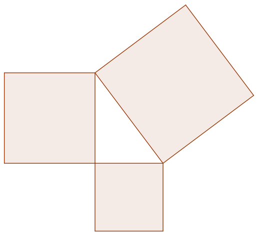
- Compass and Ruler
- Graph Paper

Instructions:

1. Use the ruler to fix the radius of a circle. You can draw any size circle you like provided it fits on the paper.
2. Count the squares inside the circle. When you are at the edge of the circle if more than half the square is inside the circle then count that square, otherwise ignore it.
3. Divide the number of squares by the square of the radius.
4. You will all get about the same answer you got last week!

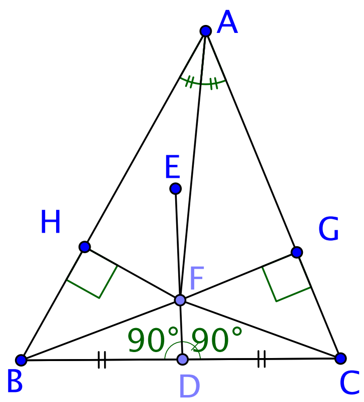
This will be unbelievable to them. You might even be accused, as I have been, of 'messing with our minds'.

The logician Raymond Smullyan described how, when he was a maths teacher in High School, he introduced the theorem of Pythagoras to his students. He would draw the diagram on the right with the squares clearly marked. He would then ask the class



> "Now suppose these three squares were made of beaten gold, and you were offered either the one large square or the two small square. Which would you choose?"

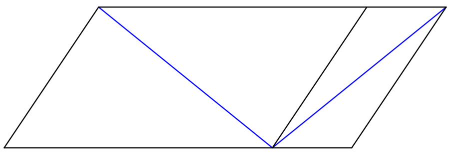
The class almost always divided 50-50 as to which was the best course of action. Both sections were equally amazed when told that it made no difference. Now it made sense to the class to try to prove the theorem. Smullyan's books, especially one called *What is the Name of this Book?* are a great source of logic puzzles.
'Prove' nonsensical results to a good class to further emphasise the need for correct proof. The following gem, which shows that every triangle is an isosceles triangle, may be well known to you but your students will not be aware of it.
Construct $[DE]$ the perpendicular bisector of $[BC]$. Construct the bisector of the angle $\angle BAC$ and let it meet $[DE]$ at F. We will consider the case where F is inside the triangle. A similar analysis applies if F is outside the triangle. Join $[FB]$ and $[FC]$. Draw $[FG]$ and $[FH]$ perpendicular to $[AC]$ and $[AB]$ respectively.



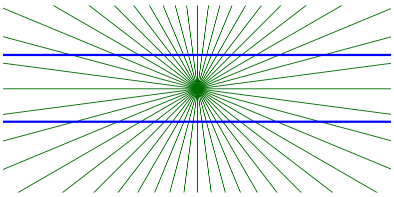
Then

$$
\begin{aligned}
\triangle AFG &= \triangle AFH \text{ (ASA)} \\
\Rightarrow |FH| &= |FG| \text{ and } |AH| = |AG| \\
\triangle FDB &= \triangle FDC \text{ (SAS)} \\
\Rightarrow |FB| &= |FC| \\
\triangle FHB &\text{ and } \triangle FGC \text{ are right-angled} \\
\Rightarrow |HB| &= |GC| \text{ (Pythagoras)} \\
\Rightarrow |AH| + |HB| &= |AG| + |GC|
\end{aligned}
$$

Hence every triangle is isosceles.
An accurate diagram will quickly show what is wrong with this 'proof' but let the students work at it. Do not tell them the fallacy too quickly.
Optical illusions again demonstrate clearly the need for proof as they show that we cannot rely on our eyes. You will find many examples on the internet and will be able to draw many in a dynamic geometry package. The reason for using dynamic geometry rather than paper is in order to demonstrate how the illusion is formed.
The drawing below shows two lines — the diagonals of two parallelograms. They do not seem to be of equal length. If you measure them you will find that they are. The illusion is called the Sander illusion after the psychologist who discovered it.



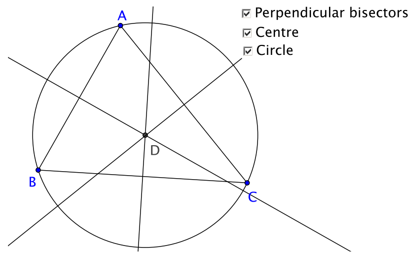
It seems that we attribute depth to the drawing and so judge the lengths badly. In the applet you can show that they are the radii of a circle and remove the illusion.
The next drawing is of two apparent curves. In fact they are parallel lines. You can see this with a ruler. In the applet the set of lines crossing them can be removed and the illusion vanishes.



Constructions and Concepts

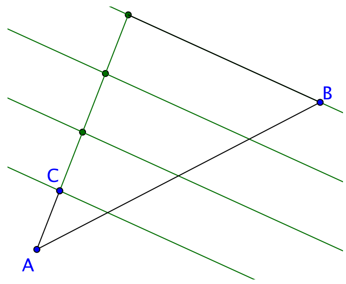
Dynamic geometry packages are very useful in constructions. The ability to change the points and lines makes the construction quite general. Students can see that it must be true. Rather than using constructions with the software in front of a class, it is a good idea to have done it beforehand, transferred to a webpage and included buttons to hide or show various steps. I will illustrate this process with two constructions. First the circumcircle of a triangle.

- Create three points
- Draw the three sides of the triangle
- Form the perpendicular bisectors of these sides
- Intersect two of these to get the circumcentre
- Draw the circumcircle
- Create three check-boxes to show and hide



Tidy up the drawing until it looks like the diagram. Export it to a webpage. In class you can now work with the webpage with no distractions such as the various windows and no danger of clicking with the wrong tool and creating a mess! The check-boxes allow you to show each step of the construction as often as necessary. The drawing is still dynamic and you can easily change the triangle. You can even show that it is possible for the circumcentre to be outside the triangle.

The next construction is Construction 7 which divides any line segment into any number of equal segments. This is quite long to set up but is worth the effort. Students, generally, can appreciate why this works in a specific case but find it difficult to generalise. The dynamic geometry helps again here. In GeoGebra the following works.

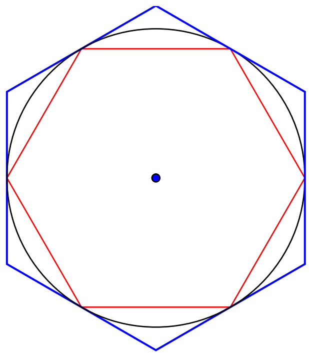


- Create a slider, n to go from 2 to 10. This will be the number of divisions.
- Create two points A and B.
- Create the segment $[AB]$.
- Create any other point, C, close to A but not on AB.

- Create the line $b = AC$.
- Create the vector, v, by typing $\texttt{v} = \texttt{Vector}[\texttt{A}, \texttt{C}]$
- Create the sequence $\texttt{points} = \texttt{Sequence}[\texttt{A} + \texttt{k}\,\texttt{v}, \texttt{k}, 1, \texttt{n}]$ which makes n equally spaced translated copies of A along the line AC.
- Create the segment $\texttt{c} = \texttt{Segment}[\texttt{Element}[\texttt{points, n}], \texttt{B}]$
(We need this for the parallel lines in the next step.)
- Create the lines $\texttt{lines} = \texttt{Sequence}[\texttt{Line}[\texttt{Element}[\texttt{points, j}], \texttt{c}], \texttt{j, 1, n}]$

You can create the usual check-boxes to show and hide whatever you want. You should hide the vector completely and talk about marking distances off with a compass. Export to a webpage and now in class you can do the construction for one particular value of n and then change n to show how everything works.

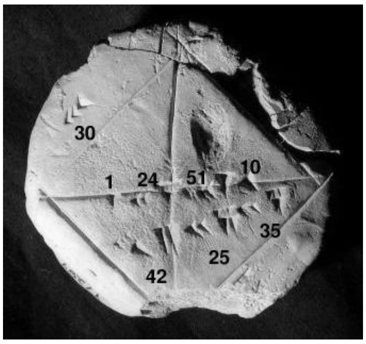
Concepts can sometimes be difficult to get across. Some children will easily imagine the increasing number of sides in the two polygons trapping the circle so that the areas all tend to the same number. Some will not. The applet showing the number of sides increasing will help here. Notice that the best value of π that we can get here taking 50 sides is between 3.1333 and 3.1457. The series converges very slowly. Later we will see how Archimedes got a better estimate. So how can we create this applet? Consider the inner polygons



- In the complex plane the n roots of 1 are spaced equally around the unit circle.
- Their values are $\cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$ as k varies from 1 to n
- So the coordinates of the vertices of the inner polygon are $\left(\cos\dfrac{2k\pi}{n}, \sin\dfrac{2k\pi}{n}\right)$
- If you create a slider n then the command $\texttt{Sequence}[(\texttt{cos}(\texttt{2kpi}/\texttt{n}), \texttt{sin}(\texttt{2kpi}/\texttt{n})), \texttt{k, 1, n}]$ will create the vertices.
- Extract these points pairwise as above to get the sides. Notice that the first side is missing. You can add it directly or let the sequence begin at 0 instead of 1. This would give two occurrences of $(1, 0)$.
- To get the exterior polygon you will need to draw a diagram and do some geometry!

Connections

Geometry has connections to all of mathematics and to much of human history. It is the most visual branch of mathematics so it should not be surprising that it seems to be the oldest form of mathematics seriously explored. The Yale Babylonian Collection holds, amongst many other fascinating tablets, a tablet from sometime between 1800 and 1600 BC which gives a value of $\sqrt{2}$ which is correct to six decimal places. The photograph at the right was taken by Bill Casselman. You can see more of his work at http://www.math.ubc.ca/∼cass/Euclid/ybc/ybc.html



The photograph contains the Babylonian symbols and the translation into the Arabic numerals we still use today. The Babylonians used 60 as the base of their number system. It is from this that we get both our units of angle and of time. The symbol along the left side of the square is $30'$. Since $1^\circ = 60'$ the square is of side $\frac{1}{2}$. We would use the theorem of Pythagoras to calculate the diagonal as $\frac{\sqrt{2}}{2}$ to get, according to my calculator, $0.7071067\ldots$. The tablet uses

$$\begin{aligned} \sqrt{2} &= 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \\ &= 1.414212962 \\ \text{which gives the diagonal} &= 0.7071064 \\ &= \frac{42}{60} + \frac{25}{60^2} + \frac{35}{60^3} \end{aligned}$$

This accuracy is better than one part in a million. How did they know? Certainly it was not by measurement as it is only in the very recent past that measurements approaching that accuracy were possible. It must have been by calculation but, unfortunately, we have no idea how.

Very little is actually known about Pythagoras. Certainly the pythagoreans were aware that $\sqrt{2}$ is irrational but the two stories about Pythagoras' reception of the news show how little we know. In one story he is so excited that he sacrifices oxen in celebration. In the other he is so appalled that he has the messenger drowned! As his view of the nature of the Universe depended on the ratios of whole numbers the second story is probably closer to the truth.
In any case we know that there are an infinite number of triplets such that

$$a^2 + b^2 = c^2$$

which links geometry to algebra. When you are creating problems for your students how can you generate as many of these as you want? For any ywo number p and q we have

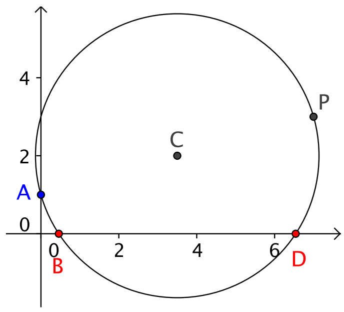
$$(p^2 + q^2)^2 = (p^2 - q^2)^2 + (2pq)^2$$

so $p^2 - q^2, 2pq$ and $p^2 + q^2$ will work. If you pick $p = 4$ and $q = 1$ then you get

$$15^2 + 8^2 = 17^2$$

Geometry is connected to algebra through coordinate geometry. While the connection to solving linear equations is well known the connection to solving quadratic equations may not be so obvious. Given a quadratic equation of the form $x^2 + ax + b = 0$ it can be solved geometrically.

The idea is to draw a circle that crosses the x-axis at the same place as the roots of the original equation. Since Euclidean geometry allows the construction of perpendiculars and the transfer of equal lengths with a compass this method uses only Euclidean techniques. It also means that any number that is the root of a quadratic equation can be constructed with straight edge and compass.



Consider the equation $x^2 + ax + b = 0$. We know from algebraic methods that the roots are

$$x = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Mark the points $A(0, 1)$ and $P(-a, b)$. Draw a circle on $[AP]$ as diameter. The centre of this circle is the point $C\left(\frac{-a}{2}, \frac{b+1}{2}\right)$. This circle has the equation

$$\left(x + \frac{a}{2}\right)^2 + \left(y - \frac{b+1}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{1-b}{2}\right)^2$$

This circle meets the x-axis when $y = 0$. Simple manipulation will show that this happens when

$$x = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

which are the roots of the original quadratic. This purely geometric method of solving a quadratic needs no hand drawn curves and is therefore much more accurate than graph plotting.

Mechanics is the study of the behaviour of physical bodies when subjected to different forces. The fundamental force that acts on everything is the force of gravity. Gravity acts on different bodies as if it were concentrated at a point. This point is called the centre of gravity. Unless an object is designed so that the centre of gravity is in a certain region it will be unstable. This is very serious. Buildings would fall. Ships would not stay above water. Planes would stall when attempting take off. It all starts, of course, with our humble triangle because the centroid of a triangle is its centre of gravity. Don't just talk about this. It is an ideal opportunity to show students how mathematics can be useful.

Requirements:

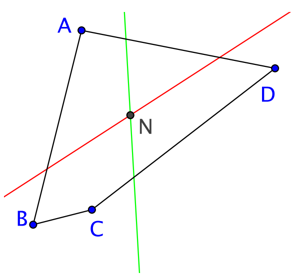
- Sheets of light cardboard.
- Compass.
- Ruler.
- Pencil.
- Scissors.

Instructions:

- Draw a triangle on the cardboard.
- Bisect each side.
- Join each vertex to the bisector of the opposite side.
- These will meet in a point (the centroid).
  Instruct the students to tell you if they do not meet. Examine any such triangles and correct the student's work.
- Cut out the triangle.
- Balance it on the point of a pencil at the centroid.

If the work has been done carefully then the students will have no difficulty balancing the cardboard. If a student, or group of students, is having difficulty then examine their work and check for accuracy.

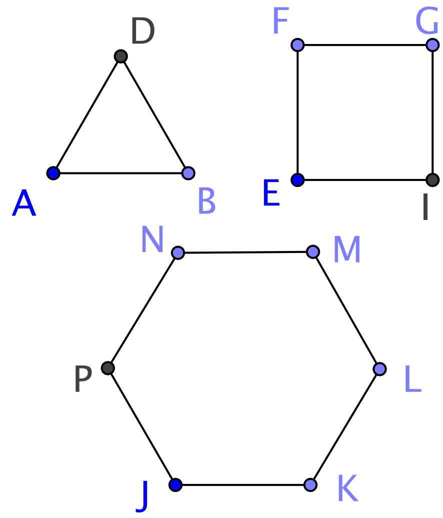
All polygons can be decomposed into triangles. This means that the same method can be used to get the centre of gravity of a polygon. Repeat the experiment with a polygon instead of a triangle. This time the teacher should draw the polygon or at least give the students an outline of the polygon to draw. In a polygon it is possible for the centre of gravity to be outside the polygon and you need to ensure that this does not happen. Four sides are enough to illustrate the idea.



Instructions:

- Divide the polygon into two triangles, say $\triangle ABC$ and $\triangle ACD$.
- Find the centroid of each triangle.
- Join the centroids with a line.
- Divide the polygon into two different triangles, say $\triangle ABD$ and $\triangle BDC$.
- Find the centroids of these triangles.
- Join them with a line.
- Mark the point where the two lines meet.
- This is the centre of gravity of the polygon.

In building various structures the shape of the triangle is very important. Use either the Geo Strips or Geomag or the equivalent to illustrate the unique structural properties of a triangle.



The equilateral triangle, when made from materials, can be broken but it cannot be distorted. Get the students to make and explore various regular polygons. They will be able to bend any of them into a different shape except for the equilateral triangle. This makes it the ideal shape for a truss in making different structures.

The square has been used in the solution of linear equations since at least the reign of Henry VIII and probably long before. The modern version of this method is called variously the Pearson square or Pearson's square. I have not been able to find the origin of this name. It is used in the paramedical professions throughout the world where, particularly, an algebraic error without a calculator could lead to very serious — if not fatal — results. It is also used to mix wines, make ice-cream and mix animal feed.
In general the Pearson square method allows an easy and visual method of solving simultaneous equations of the form

$$\begin{aligned} x + y &= m \\ ax + by &= n \end{aligned}$$

where a, b, m and n are constants. In the real world of farming, ice-cream making or Pharmacy these numbers can be very awkward. For example, in a problem involving making up a hydrocortisone ointment, the equations might be

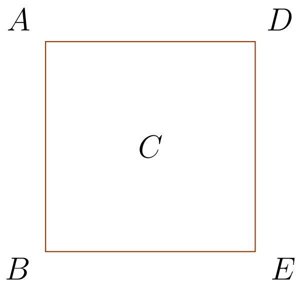
$$\begin{aligned} x + y &= 100 \\ 0.01x + 0.045y &= 2.5 \end{aligned}$$

which would be solved using algebra to get

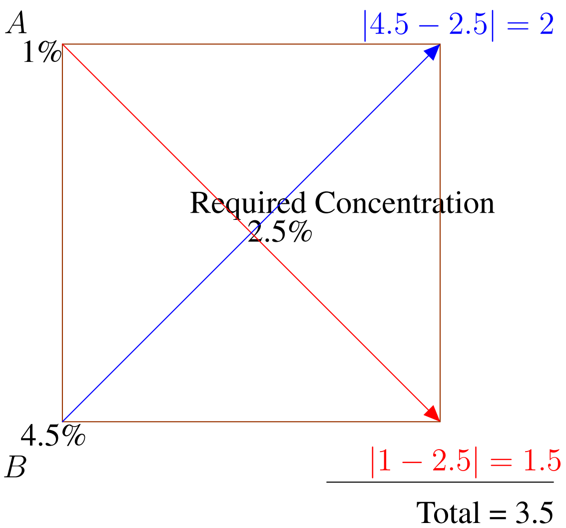
$$\begin{aligned}\Rightarrow y &= \frac{300}{7}\\ \Rightarrow x &= \frac{400}{7}\end{aligned}$$

This type of calculation may be only algebra but a large number of people find it very difficult to do. The US army insists on its medical personnel learning the old methods since it is quite likely for them to be in a situation without access to a computer or calculator.

The diagram on the right shows a square with four labelled corners and a labelled point in the centre. The points A and B are quantities that are given (for example hydrocortisone ointments of concentrations 1% and 4.5%). The number C is the required value (for example an ointment of concentration 2.5%). The number D is calculated as $|B - C|$ and E is calculated as $|A - C|$.



The numbers D and E are the required amounts of the substances A and B to use. Their sum gives the total amount which can be converted to a percentage and then made up. All of this may seem a bit abstract but in fact it is very easy to use in practice as the example below shows.



The amount of A to use is $\dfrac{2 \times 100}{3.5} = \dfrac{400}{7}$ while the amount of ointment B is $\dfrac{1.5 \times 100}{3.5} = \dfrac{300}{7}$. This could well be an example of geometry saving your life.

Transformation geometry is used to move images in making cartoons and CGI in films. Using GeoGebra it is easy to get any of the transformations that your students will study. You can make it more alive for them if you import graphics and move these. It will demonstrate to them how transformation geometry is used in the world of entertainment. The internet has many sites that contain royalty free cartoons which you can use in class to enliven the study of transformations.

We want to show the cartoon at the right being moved through a central symmetry. The same method can be used for any of the transformations. It is easy to create a point and use the Reflect Tool to get the image of the cartoon. It is more effective to show the transformation as a dynamic movement. You will be able to do this using the following method.

© www.ClipProject.info

Instructions:

1. Create three points A, B and C where A and B are the bottom left and right corners of a rectangle respectively. C will be the top left corner.

2. Create a point, O, as the centre of symmetry.

3. Reflect the points A, B and C in O to get A', B' and C'.

4. Create a slider, t, to go from 0 to 1 in steps of .01. You can change the increment later if you want to.

5. Use the Input window to create the three points
$\mathtt{A''} = \mathtt{t\,A'} + (\mathtt{1} - \mathtt{t})\,\mathtt{A}$
$\mathtt{B''} = \mathtt{t\,B'} + (\mathtt{1} - \mathtt{t})\,\mathtt{B}$
$\mathtt{C''} = \mathtt{t\,C'} + (\mathtt{1} - \mathtt{t})\,\mathtt{C}$

6. Import the cartoon.

7. Go to Object Properties.

8. Click on the Position tab.

9. Attach Corner 1 to A'', Corner 2 to B'' and Corner 4 to C''.

References

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- *Journey through Genius* William Dunham
- *Introduction to Geometry* H. S. M. Coxeter
- `http://www.ted.com/`
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